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ON CONFORMAL CORRESPONDENCE OF SURFACES AND MANIFOLDS.*

By P. SAMUEL.

I. Introduction.

In the ordinary euclidean 3-space the problem of finding two surfaces in conformal correspondence with parallel corresponding tangent planes was solved by Christoffel.[1] The method is the consideration of the linear correspondence between the corresponding vectors of the tangent planes: one takes the principal directions of this linear operator as coordinate directions. The result is: either the two surfaces are minimal, or their lines of curvature form an isothermic net, according as the principal directions are isotropic (the linear correspondence being a proper similarity) or real (the linear correspondence being a proper similarity followed by a symmetry).

This method of taking the principal directions of a suitable linear operator as coordinate directions is very often used in differential geometry: for example the consideration of the lines of curvature.

In this paper we shall study the general case of two n-manifolds in an euclidean N-space, in conformal correspondence, the tangent n-planes at two corresponding points being parallel. We shall use the same general method: take the principal directions of the linear operator between the corresponding vectors in the tangent n-planes as coordinate directions. For convenience this operator will be called "parallel tangent planes operator" or "the operator."

After having fixed the notations and the terminology used, I establish in II and III some properties of the operator which will be used throughout this paper. IV is a rapid review of some important kinds of manifolds in the euclidean N-space. V studies the case of 2-dimensional manifolds; the results obtained are a mere generalization of those of Christoffel: we obtain as only solutions the minimal surfaces and the surfaces which have isothermic lines of curvature. With manifolds of any dimension the case where all the principal directions of the operator are real can be studied completely; this

* Received December 9, 1946.

[1] Christoffel, E. B., "Über einige allgemeine Eigenschaften der Minimumsflächen," *Journal für die reine und angewandte Mathematik*, vol. 67 (1867), p. 218. See also: Darboux, "*Théorie des surfaces*," t. II, livre IV, ch. XI, pp. 239-55, 1889.

is done in VI; the result is that the manifolds must belong to a very restricted class; an application is given to the hyperspherical representation of a hypersurface: when the space is of dimension > 3 it is only conformal for the hypersphere itself and the generalization of the catenoid. In VII is studied the case of 3-dimensional manifolds: if the principal directions of the operator are holonomic we give a complete solution, that is parametric representations of the two manifolds in terms of 3 arbitrary vector functions of one variable; in the non-holonomic case we get only the ds^2 of the manifolds in terms of an arbitrary function of two variables and some necessary and sufficient conditions on the second quadratic forms. VIII is the study of the general case, with the restriction that the principal directions are holonomic; a complete answer is given: the only possible manifolds are essentially: the translation manifolds generated by two totally isotropic manifolds and their cartesian products,—and a generalization of the manifolds obtained in VII. IX is the study of the case where the correspondence between the two manifolds is not only conformal but isometric; using classical lemmas about geodesics and geodesic coordinates, we prove that the linear correspondence between the two parallel tangent n-planes (which is a rotation) has its angles constant when decomposed in its canonical form; after that we consider the case of 3- and 4-dimensional manifolds, and we prove the holonomy of the principal directions; we are therefore reduced to the case studied in the preceding section.

It should be noted that our considerations are only *local* and restricted to *analytic manifolds.*

Notations and terminology.

In this paper we use the notations and the terminology of J. A. Schouten.[2] $\partial_\alpha x$ denotes the ordinary derivative with respect to the variable u^α, or the non-holonomic derivative with respect to the coordinate direction of index α; $\Gamma^\gamma{}_{\alpha\beta}$ are the Christoffel symbols defining the affine connection of the manifold considered as a Riemannian space; $g_{\alpha\beta}$ is the metric tensor; ∇_α is the covariant derivation with respect to the affine connection; $\partial_{\alpha\beta}$ and $\nabla_{\alpha\beta}$ denote the second derivatives in the order β, α. $\Omega^\gamma{}_{\alpha\beta}$ is the "non-holonomy object" relative to our coordinate directions, which are in the general case non-holonomic; this symbol expresses the "deviation of the commutativity" of the second derivatives in terms of the first ones $\partial_{\alpha\beta} - \partial_{\beta\alpha} = \Omega^\gamma{}_{\beta\alpha}\partial_\gamma$. We use the classical summation convention.

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[2] Schouten, J. A., "*Einführung in die neueren Methoden der Differentialgeometrie,*" Groningen, 1935.

The corresponding points on the two manifolds will be denoted by M and P, which will denote also the vectors $\vec{OM}$ and $\vec{OP}$, O being the origin in the euclidean N-space. The two manifolds will be represented in terms of the same (holonomic or not) parameters u^α, two corresponding points being obtained for the same values of the parameters, as usual. Some quantities, like the $\Omega^\gamma{}_{\alpha\beta}$, are the same for the two manifolds; for the others we must use a subscript: $g_{M,\alpha\beta}$, $g_{P,\alpha\beta}$, $\Gamma_M{}^\gamma{}_{\alpha\beta}$, $\Gamma_P{}^\gamma{}_{\alpha\beta}$.

II. General Properties of the Operator.

We denote by $\{\vec{n_a}\}$ a free system of $N-n$ vectors orthogonal to the tangent n-planes at M (or P). The Frenet-formulae of the manifold (M) are:

$$\partial_{\alpha\beta}M = \Gamma_M{}^\gamma{}_{\alpha\beta}\cdot\partial_\gamma M + a_{M\alpha\beta^a}\cdot\vec{n_a}$$

(The $a_{\alpha\beta^a}du^\alpha du^\beta$ are the second fundamental forms of the manifold (M) imbedded in the euclidean N-space.)

We define the operator $b_\alpha{}^\beta$ by the relation $\partial_\alpha P = b_\alpha{}^\beta\cdot\partial_\alpha M$. If the tensor $b_\alpha{}^\beta$ is in diagonal form, the coordinate directions being its principal directions, then $b_\alpha{}^\beta = {}_a s\cdot\delta_\alpha{}^\beta$, and $\partial_\alpha P = {}_a s\cdot\partial_\alpha M$ (α not summed).

We consider the compatibility conditions

$$\partial_{\beta\alpha}P - \partial_{\alpha\beta}P = \Omega^\gamma{}_{\alpha\beta}\cdot\partial_\gamma P = {}_\gamma s\cdot\Omega^\gamma{}_{\alpha\beta}\cdot\partial_\gamma M = \partial_\beta({}_a s)\partial_a M - \partial_a({}_\beta s)\cdot\partial_\beta M + {}_a s\cdot\partial_{\beta a}M - {}_\beta s\cdot\partial_{\alpha\beta}M.$$

Or

$${}_\gamma s\cdot\Omega^\gamma{}_{\alpha\beta}\cdot\partial_\gamma M = \partial_\beta({}_a s)\cdot\partial_a M - \partial_a({}_\beta s)\cdot\partial_\beta M + ({}_a s\Gamma_{M\beta a}{}^\gamma - {}_\beta s\cdot\Gamma_{M\alpha\beta}{}^\gamma)\partial_\gamma M + ({}_a s - {}_\beta s)a_M{}^a{}_{\alpha\beta}n_a.$$

From these vector equalities we conclude

a) $$({}_a s - {}_\beta s)a_M{}^a{}_{\alpha\beta} = 0.$$

Two principal directions relative to two distinct characteristic roots are conjugate with respect to all the second forms.

b) If γ is different from α and β

(1) $${}_\gamma s\cdot\Omega^\gamma{}_{\alpha\beta} = {}_a s\cdot\Gamma_M{}^\gamma{}_{\beta a} - {}_\beta s\cdot\Gamma_M{}^\gamma{}_{\alpha\beta}.$$

c) If $\gamma = \alpha$ (same formula if $\gamma = \beta$)

(2) $${}_a s\cdot\Omega_{(a)\beta}{}^{(a)} = \partial_\beta({}_a s) + {}_a s\cdot\Gamma_{M\beta(a)}{}^{(a)} - {}_\beta s\cdot\Gamma_{M(a)\beta}{}^{(a)}.$$

The parenthesis (α) denotes that there is no summation with respect to α.

III. Principal Directions when the Correspondence is Conformal.

In this case $ds_P^2 = A^2 ds_M^2$. Or $g_{P\alpha\beta} = A^2 g_{M\alpha\beta} = {}_\alpha s_\beta s g_{M\alpha\beta}$. Suppressing the subscript M we have

$$(3) \qquad ({}_\alpha s \cdot {}_\beta s - A^2) g_{\alpha\beta} = 0.$$

We conclude from (3):

1) If $\alpha = \beta$:

a) $g_{\alpha\beta} = 0$. We obtain $n - p$ isotropic principal directions, complex conjugate by pairs, denoted here by latin indices: $g_{ii} = g_{i^*i^*} = 0$.

b) $g_{\alpha\alpha} \neq 0$ (real principal directions). Then ${}_\alpha s = \pm A$. The q directions with ${}_\alpha s = +A$ will be denoted by $\alpha, \beta, \gamma \cdots$ (positive directions), and the $p - q$ directions with ${}_\alpha s = -A$ by $\lambda, \mu, \nu \cdots$ (negative directions).

2) For two positive or two negative directions, (3) is verified. For one positive and one negative one must have $g_{\alpha\lambda} = 0$.

3) g_{ii^*}, sum of products of complex conjugate quantities, is real > 0; hence ${}_i s_{i^*} s - A^2 = 0$; ${}_i s = Ae^{i\theta_i}$; ${}_{i^*} s = Ae^{-i\theta_i}$ $(\theta_i \neq k\pi)$.

4) Then one sees easily that $g_{i\alpha} = g_{i\lambda} = 0$, and $g_{ik} = 0$, if $\theta_i + \theta_k \neq 2k\pi$.

IV. Some Classical Manifolds of the Euclidean N-Space.

In this section we shall review a certain number of manifolds imbedded in the euclidean N-space. These manifolds are classical and the only aim of this short review is to give a name to each of them in order to increase the brevity of the rest of this paper.

1) Translation manifolds.

These manifolds can be represented parametrically by

$$M = U(u^\alpha) + V(v^\lambda).$$

The vector U depends only on the variable u^α, V only on the other variables v^λ.

2) Particular case: "cylinder."

In many questions the vectors $\partial_\alpha M$ and $\partial_\lambda M$ must be orthogonal: $\partial_\alpha U \cdot \partial_\lambda V = 0$. They vary independently. If $\mathfrak{U}$ and $\mathfrak{V}$ are the linear subspaces spanned respectively by them, $\mathfrak{U}$ and $\mathfrak{V}$ are totally orthogonal. Then,

up to constant vectors, $U \varepsilon \mathfrak{U}$ and $V \varepsilon \mathfrak{V}$. In this case it may be useful to take the first axis of cartesian coordinates in $\mathfrak{U}$, and the last ones in $\mathfrak{V}$.

3) Revolution manifolds.

In the N-space there are several kinds of revolution manifolds according as the rotation takes place around an $(N-2)$-plane, an $(N-3)$-plane, $\cdots$. In the first case, with suitable axis, the parametric representation of the manifold is

$$x_1 = r \cdot \cos\theta\,;\ x_2 = r \cdot \sin\theta\,; \qquad x_j = x_j(r),\ 2 < j \leq N.$$

In the second case

$$x_1 = r \cdot \cos\theta \cdot \cos\phi\,;\ x_2 = r \cdot \cos\theta \cdot \sin\phi\,;\ x_3 = r \cdot \sin\theta\,;\ x_j = x_j(r),\ 3 < j.$$

4) Totally isotropic manifolds.

A manifold is called totally isotropic if every tangent vector is isotropic (i.e. of length zero). Since $dM = \partial_\alpha M \cdot du^\alpha$, $(dM)^2 = 0$ implies $\partial_\alpha M \cdot \partial_\beta M = 0$: the coordinate directions must be isotropic and mutually orthogonal. This necessary condition (C) is evidently sufficient.

If we take the first n coordinates x^a in the euclidean N-space as parameters on the manifold, the others being denoted by $x^a, x^b \cdots$ the condition (C) is:

$$1 + \sum_{a=1}^{a=N-n} (\partial_a x^a)^2 = 0\,; \qquad \sum_{a=1}^{a=N-n} \partial_a x^a \partial_\beta x^a = 0.$$

If $x^a = iy_a$, these conditions express that the n-manifold of the $(N-n)$-space, with cartesian coordinates y_a and parameters x^a, has

$$ds^2 = (dx^a)^2 + (dx^\beta)^2 + \cdots.$$

The study of the totally isotropic manifolds is therefore equivalent to the study of the manifolds with an euclidean ds^2. Note that we must have $N - n \geq n$, or $n \leq N/2$: in an N-space a totally isotropic manifold is at most of dimension $N/2$. Those of dimension $N/2$ are linear, the manifold with euclidean ds^2 being then the whole $(N-n)$-space.

5) Translation manifolds generated by two complex conjugate totally isotropic manifolds.

A totally isotropic manifold is always imaginary. Then we consider the translation manifold generated by two complex conjugate totally isotropic manifolds, the real points being obtained by associating two complex conjugate

points: $M = P(u^\alpha) + P^*(u^{\alpha^*})$. $g_{\alpha\beta} = g_{\alpha^*\beta^*} = 0$; $g^{\alpha\beta} = g^{\alpha^*\beta^*} = 0$. Since $\partial_{\alpha\beta^*} M = 0$, $a_{\alpha\beta^*}{}^{i} = 0$ for all the second forms; $a_\alpha{}^{\beta,i} = g^{\alpha\lambda} a_{\lambda\beta}{}^{i} + g^{\alpha\lambda^*} a_{\lambda^*\beta}{}^{i} = 0$, and $a_{\alpha^*}{}^{\beta^*,i} = 0$. The characteristic equation of the matrix $(a_\alpha{}^{\beta,i})$ is;

$$\begin{vmatrix} s & 0 & . & . \\ 0 & s & . & . \\ . & . & s & 0 \\ . & . & 0 & s \end{vmatrix} = 0.$$

Applying Laplace's rule, we see, two corresponding minors of order $n/2$ having as factor the same power of s, that the characteristic equation will contain only terms with an even power of s. Therefore the roots of the equation, which are *the principal curvatures relative to the second form* $a_{\alpha\beta}{}^{i}\, du^\alpha du^\beta$, *are opposite by pairs;* their sum, *the mean curvature normal vector, is zero.* With two isotropic curves in the 3-space we obtain the *minimal surfaces.*

V. Two-Dimensional Case.

In this case any net of coordinate lines is holonomic. The two parameters on the surfaces will be called u and v.

1) Two positive principal directions (or two negative ones).

$$\partial_u P = A \partial_u M, \qquad \partial_v P = A \partial_v M.$$

The compatibility condition is $\partial_{uv} P - \partial_{vu} P = \partial_u A \cdot \partial_v M - \partial_v A \cdot \partial_u M = 0$. Therefore: $\partial_u A = \partial_v A = 0$, $A = \text{Const.}$ *The surfaces are homothetic.*

2) Two isotropic principal directions.

$$\partial_u P = A e^{i\theta} \cdot \partial_{ii}, \qquad \partial_v P = A \cdot e^{-i\theta} \cdot a_v M.$$

The compatibility condition is

$$\partial_v (A e^{i\theta}) \cdot \partial_u M - \partial_u (A e^{-i\theta}) \cdot \partial_v M + 2Ai \cdot \sin\theta \cdot \partial_{uv} M = 0.$$

But

$$(\partial_u M)^2 = (\partial_v M)^2 = 0, \qquad \partial_u M \cdot \partial_v M \neq 0.$$

By scalar multiplication of the compatibility condition by $\partial_u M$ and $\partial_v M$, and since $\partial_{uv} M \cdot \partial_u M = 1/2 \cdot \partial_v (\partial_u M)^2 = 0$, one obtains $\partial_v (A e^{i\theta}) = \partial_u (A e^{-i\theta}) = 0$. Hence (if $\sin\theta = 0$ we are in the first case) $\partial_{uv} M = 0$. Therefore

$$M = U(u) + V(v), \qquad U'^2 = V'^2 = 0.$$

The surfaces are translation surfaces generated by two isotropic curves. Then $P = U_1(u) + V_1(v)$, the curves U and U_1, V and V_1 having parallel tangents at corresponding points.

3) One positive and one negative principal direction.

These directions are orthogonal (III) and conjugate (II). Therefore the surfaces have lines of curvature, which are taken as coordinate lines.

$$ds^2 = E \cdot du^2 + G \cdot dv^2, \qquad ds^2 = A(E \cdot du^2 + G \cdot dv^2)$$

$$\partial_u P = A \cdot \partial_u M, \qquad \partial_v P = -A \cdot \partial_v M.$$

The compatibility condition is

$$2A \cdot \partial_{uv} M + \partial_u A \cdot \partial_v M + \partial_v A \cdot \partial_u M = 0.$$

By scalar multiplication by $\partial_u M$ and $\partial_v M$ we get

$$A \cdot \partial_v E + E \cdot \partial_v A = 0 \text{ or } E = f(u)/A$$

$$A \cdot \partial_u G + G \cdot \partial_u A = 0 \text{ or } G = g(v)/A.$$

Changing the variables u and v one can take $E = G = 1/A$.

$$ds^2_M = (1/A) \cdot 2(du^2 + dv^2).$$

The lines of curvature are isothermic. Conversely the compatibility condition is verified in this case.

Therefore, to each surface having lines of curvature which are isothermic corresponds another surface with the same property; these two surfaces are in conformal correspondence with parallel tangent planes.

Example: revolution surfaces.

$$x_1 = r \cdot \cos\theta, \quad x_2 = r \cdot \sin\theta, \quad x_j = x_j(r) \qquad (j \geq 3).$$

The following vectors are $N - 2$ linearly independent normal vectors:

$$(-x'_j \cos\theta, -x'_j \sin\theta, 0, \cdots 0, 1, 0, \cdots, 0).$$

The ds^2 is:

$$ds^2 = (1 + \Sigma_{j \geq 3} x'^2_j) \cdot dr^2 + r^2 \cdot d\theta^2.$$

The $N - 2$ second forms relative to the preceding normal vectors are:

$$(x'_j{}' \, dr^2 + r \cdot x'_j \cdot d\theta^2)/\sqrt{1 + x'^2_j}.$$

They are reduced simultaneously to sums of squares; hence there exist

lines of curvature which are the meridians and the parallels. Since $ds^2 = r^2(du^2 + d\theta^2)$, where $du = \sqrt{1 + \Sigma_{j\geq 3} x'^2_j(r)} \cdot dr/r$, they are isothermic.

If the coordinates of P are $(y_1, y_2, \cdots, y_N)$, the integration of

$$\partial_r P = 1/r^2 \cdot \partial_r M, \qquad \partial_\theta P = -1/r^2 \cdot \partial_\theta M,$$

gives

$$y_1 = -\cos\theta/r, \; y_2 = -\sin\theta/r, \; y_j = \int dx_j/r^2 = \int x'_j(r)\,dr/r^2 \qquad (j \geq 3).$$

In order to avoid the integrations one can write

$$x_j = r(\alpha_j - r\alpha'_j), \qquad y_j = -(\alpha_j + r\alpha'_j)/r$$

α_j being an arbitrary function of r only.

In the case of ordinary space one obtains an interesting correspondence between the meridians, i. e., between plane curves. To the curve $(z(t), r(t))$ corresponds the curve $(\int dz/r, -1/r)$. We take z as the independent variable, and if we denote by R the radius of curvature of the meridian, by N the length of the normal to the meridian limited by the incidence point and the z-axis,

$$R = (1 + r'^2)^{3/2}/r'', \qquad N = r(1 + r'^2)^2, \qquad R/N = (1 + r'^2)/rr'.$$

For the other curve; $r_1 = -1/r$, $dz_1 = dz/r$.

$$r'_1 = dr_1/dz_1 = \frac{1}{r^2}\,\frac{dr}{dz}\,\frac{dz}{dz_1} = (1/r^2)r'r^2 = r' \qquad \text{(the tangents are parallel)}$$

$$r''_1 = dr'_1/dz_1 = \frac{dr'}{dz}\,\frac{dz}{dz_1} = r''r^2.$$

Hence $R_1/N_1 = -R/N$.

At corresponding points the quantities R/N are opposite.

To a circle with center on Oz ($N = R$; sphere in the space) corresponds the catenary ($N = -R$; minimal surface of revolution in the space).

To the cycloid with base Oz ($R = 2N$) corresponds the parabola with directrix Oz ($R = -2N$).

There are other interesting couples of curves:

Hyperbola $(z = a \cdot \sinh t, \; r = b \cdot \cosh t)$ and sinusoid $(r_1 = -(1/b) \cdot \sin b^2 z_1/a)$.

Hyperbola $(z = a \cdot \cosh t, \; r = b \cdot \sinh t)$ and $(r = -(1/b) \cdot \sinh b^2 z_1/a)$.

Circle tangent to Oz $(z = a \cdot \sin t, \; r = a + \cos t)$ and rectifiable cubic of Tschirnhausen

$(z_1 = -(1/6a)(3u - u^3),\ r_1 = -(4/6a) - (1/6a)(3u^2 - 1);$
$u = \tan t/2).$

Parabola with axis Oz $(z = r^2/2p)$ and exponential curve $(r_1 = -e^{pz_1})$.

VI. Case where all the Principal Directions are real.

1) The formula of II b) with $\gamma = \lambda$, ${}_\gamma s = -A$, ${}_\alpha s = {}_\beta s = A$, gives $-A \cdot \Omega^\lambda{}_{\alpha\beta} = A(\Gamma_M{}^\lambda{}_{\beta\alpha} - \Gamma_M{}^\lambda{}_{\alpha\beta}) = A\Omega^\lambda{}_{\alpha\beta}$. Therefore $\Omega^\lambda{}_{\alpha\beta} = 0$, and for the same reason $\Omega^\alpha{}_{\lambda\mu} = 0$. Hence the two linear systems $\{\partial_\alpha f = 0\}$ and $\{\partial_\lambda f = 0\}$ are complete. One can find *holonomic coordinate directions* such that the first ones (α) are principal directions relative to A, the others (λ) principal directions relative to $-A$.

2) The formula of II c) gives, if there exist at least two positive directions α, β: $A\Omega^{(\alpha)}{}_{(\alpha)\beta} = \partial_\beta A + A \cdot \Omega^{(\alpha)}{}_{(\alpha)\beta}$, or $\partial_\beta A = 0$. In the same way $\partial_\lambda A = 0$.

Therefore A is a constant unless one direction is alone of its kind.

In the case $n = 2$, already studied, the two directions could be of different kinds, and A could depend on the two variables. In the case $n \geq 3$, what is a little surprising at first sight, we have less freedom. If one direction is alone of its kind, the other ones are in number ≥ 2, and A can depend only on the variable corresponding to the first direction.

Case A constant.

If all the directions are positive (or negative) the two manifolds are homothetic. We suppose now that we have

$$\partial_\alpha P = A \cdot \partial_\alpha M, \qquad \partial_\lambda P = -A \cdot \partial_\lambda M.$$

Integrating:

$$P = A(M + \Phi(u^\lambda)), \qquad P = A(-M + \Psi(u^\alpha)).$$

Hence

$$P = \tfrac{1}{2}A(\Psi(u^\alpha) + \Phi(u^\lambda))$$
$$M = \tfrac{1}{2}(\Psi(u^\alpha) - \Phi(u^\lambda)).$$

But we must have also $\partial_\alpha M \cdot \partial_\lambda M = 0$. Applying IV 2) we see that the vectors Φ and Ψ must vary in two totally orthogonal subspaces.

Case A non-constant.

We may suppose that $n - 1$ directions (u^α) are positive and the last one (v) negative. We may take $A = v$ since A depends only on v.

$$\partial_a P = v \cdot \partial_a M, \qquad \partial_v P = -v \cdot \partial_v M.$$

The integration of the first gives $P = vM + V(v)$. Then the last one may be written $2v\partial_v M + M + V'(v) = 0$. The integration of this linear differential equation of the first order gives

$$M = 1/\sqrt{v}(M_1(u^a) - \int (V'(v)dv/2\sqrt{v}))$$

$$P = \sqrt{v}(M_1(u^a) - \int (V'(v)dv/2\sqrt{v})) + V(v).$$

But one must have $\partial_a M \cdot \partial_v M = 0$, or $\partial_a M_1 \cdot (M + V') = 0$, or

$$(1) \qquad M_1 \cdot \partial_a M_1 + \partial_a M_1 \cdot (V'\sqrt{v} - \int (V'dv/2\sqrt{v})) = 0.$$

If we write $V_1 = V'\sqrt{v} - \int (V'dv/2\sqrt{v})$, we get $\partial_a M_1 \cdot \partial_v V'_1 = 0$.

Then, by IV 2), $M_1 = M_0 + M_2$, $V_1 = V_0 + V_2$, M_0 and V_0 being constant vectors, $M_2(u^a)$ and $V_2(v)$ vectors in two totally orthogonal subspaces $\mathfrak{M}$ and $\mathfrak{B}$. (1) may be written then $\partial_a M_2 \cdot (M_0 + V_0 + M_2) = 0$. One may suppose $M_0 = 0$ (it may be absorbed by V) and $V_0 \in \mathfrak{M}$. Then $V_0 + M_2$ has a constant length. Let $U(u^a)$ denote a unit vector of $\mathfrak{M}$, $M = k \cdot U(u^a) - V_0$, k being a constant.

Let us consider $V_0 + V_2 = V'\sqrt{v} - \int (V'dv/\sqrt{v})$. The projection on $\mathfrak{B}$, since V_2 may vary, does not impose any conditions on the components of V' in $\mathfrak{B}$. But, by projection on $\mathfrak{M}$, the vector W', the projection of V' on $\mathfrak{M}$, must satisfy $W'\sqrt{v} - \int W'dv/\sqrt{v} = \text{const}$. Hence W' is a constant vector W_0. Then $V' = V'_3 + W_0$ ($V'_3 \in \mathfrak{B}$, $W_0 \in \mathfrak{M}$). In order to avoid the integrations, we take $1/\sqrt{v} = w$ as a new variable, and for V a derivative W' ($W \in \mathfrak{B}$). M may be absorbed by W. Writing $-M_0$ instead of V_0, we get the following parametric representations:

$$M = w(kU(u^a) + M_0) + w(W - W'w)$$

$$P = (1/w)(kU(u^a) + M_0) + (1/w)(W - W'w)$$

where $U(u^a)$ is a unit vector in $\mathfrak{M}$, $M_0 \in \mathfrak{M}$, k is a numerical constant, and $W \in \mathfrak{B}$ subspace orthogonal to $\mathfrak{M}$.

These formulae are a generalization of the formulae for the correspondence between revolution surfaces.

Conformal hyperspherical representation of a hypersurface.

If the manifold (M) is a hypersurface, it can be mapped by parallel tangent planes on the hypersphere of radius 1. Then the "operator" is the

second fundamental tensor $a^{\alpha}{}_{\beta}$, the principal directions are the tangents to the lines of curvature, and the characteristic roots ${}_{\alpha}s$ are the principal curvatures.

If this correspondence is conformal, all the principal curvatures must be equal in absolute value. We have here an example of decomposition of properties. In the classical case of a surface in the 3-space the condition of conformal correspondence with its spherical image is that the surface be minimal. But the reason for this fact is not "the sum of the principal curvatures is 0," but "the principal curvatures have the same absolute value." These two properties, indentical in the 3-space, become different in higher spaces. Here is one of the principal interests of the generalizations; to show the real reason for the properties, and to separate what is natural from what is due to coincidence.

For a hyperspherical representation the principal directions are real. Since the hypersphere is not a translation manifold, the case "A constant" gives only the trivial homothety. In the case "A non constant" $U(u^{\alpha})$ must run over all the unit vectors of an $(N-1)$-dimensional subspace, for convenience the subspace spanned by the x_j axes, $j \geq 2$. Then in the parametric representation, $M_0 = 0$, $w = \Sigma_{j\geq 2} x^2{}_j$ (k gives only a homothety, and we take $k = 1$). The other coordinate $x_1 = w(W - W'w)$ (W being here a scalar function of w) must satify ${x_1}^2 + w^2 = 1$, or $W'w - W = \sqrt{1-w^2}/w$. Taking $1/w = \cosh \phi$, the solution is: $W = -(\sinh 2\phi - 2\phi + a)/4 \cosh \phi$. The locus of P is therefore a *revolution manifold,* whose meridian has the following parametric representation.

$$r = \Sigma_{j\geq 2} x^2{}_j = 1/w = \cosh \phi, \qquad x_1 = (W'w + W) \cdot 1/w = \phi - (a/2).$$

This meridian is the *catenary* $r = \cosh(x_1 + a/2)$, and the manifold is the natural generalization of the minimal surface of revolution.

We have here a second example of a loss of freedom by passage from ordinary surfaces to higher manifolds: the surfaces with conformal, non-homothetic spherical representations are all the minimal surfaces, which depend on an arbitrary analytic function. For the hypersurfaces one obtains essentially one solution.

We may compare this fact to the conformal mappings of an euclidean space itself. For the plane the solutions depend on an arbitrary analytic function; but for the n-space ($n \geq 3$) the solution is given by the anallagmatic group which depends on a finite number of parameters, $(n+1)(n+2)/2$.[3]

[3] Klein, F., "*Vorlesungen über höhere Geometrie,*" p. 197, Berlin, 1926.

VII. Three-Dimensional Case.

There still remains the case where one principal direction is real (we may take it positive without loss of generality) and the two others are isotropic.

Holonomic case.

Calling u, v, w the parameters one must have

$$\partial_u P = A\partial_u M, \qquad \partial_v P = A \cdot e^{i\theta}\partial_v M, \qquad \partial_w P = A \cdot e^{-i\theta}\partial_w M$$

and

$$ds^2 = E \cdot du^2 + 2H \cdot dv \cdot dw.$$

The compatibility conditions are

$$A(1 - e^{-i\theta})\partial_{uw}M = \partial_u(Ae^{-i\theta})\partial_w M - \partial_w A\partial_u M.$$

$$A(1 - e^{i\theta})\partial_{uv}M = \partial_u(Ae^{i\theta})\partial_v M - \partial_v A \cdot \partial_u M.$$

$$2Ai \cdot \sin\theta \cdot \partial_{vw}M = \partial_v(Ae^{-i\theta})\partial_w M - \partial_w(A^{i\theta})\partial_v M,$$

By scalar multiplication with $\partial_u M$, $\partial_v M$, $\partial_w M$, and since

$$\partial_u M \cdot \partial_v M = \partial_u M \cdot \partial_w M = (\partial_v M)^2 = (\partial_w M)^2 = 0, \qquad (\partial_u M)^2 = E,$$

$\partial_v M \cdot \partial_w M = H \neq 0$, we obtain two identities and

$$\partial_v(Ae^{-i\theta}) = \partial_w(Ae^{i\theta}) = \partial_u(Ae^{i\theta}) = \partial_u(Ae^{-i\theta}) = \partial_u H = 0$$

$$\partial_w E/E = -2(\partial_w A/A(1 - e^{-i\theta})); \qquad \partial_v E/E = -2(\partial_v A/A(1 - e^{i\theta})).$$

Changing the variables v and w, we can take $Ae^{-i\theta} = w^2$, $Ae^{i\theta} = v^2$. Then $A = vw$, $e^{i\theta} = v/w$, $e^{-i\theta} = w/v$; and the integration of the two conditions for E gives $E = \Psi_1(u, v)(w - v/w)^2 = \Psi_2(u, w)((w - v)/v)^2$. Comparing them; $E = \phi(u)(v - w)^2/v^2w^2$. Changing the variable u we can take $E = (v - w)^2/v^2w^2$, $H = H(v, w)$. Then the compatibility conditions take the simpler form

$$\partial_{uw}M = (v/w(w - v))\partial_u M, \qquad \partial_{uv}M = (w/v(v - w))\partial_u M, \qquad \partial_{vw}M = 0$$

and are solved by $M = ((w - v)/vw)U(u) + V(v) + W(w)$.

It remains to study the conditions for ds^2_M; for E it is sufficient to take $U'^2 = 1$. The conditions $\partial_u M \cdot \partial_v M = \partial_u M \cdot \partial_w M = 0$ become

$$-(1/v^2)UU' + U'V' = +(1/w^2)UU' + U'W' = 0.$$

If

$$U^2 = \Psi(u), \qquad UU' = \tfrac{1}{2}\Psi'(u),$$

and

$$(U'/\Psi'(u)) \cdot (2v^2V') = 1, \qquad (U'/\Psi'(u))\,(2w^2W') = -1.$$

If

$$U'/\Psi'(u) = U_1(u), \qquad 2v^2V' = V_1(v), \qquad 2w^2W' = W_1(w),$$

we obtain by derivation

$$U'_1V_1 = 0, \qquad U_1V'_1 = 0, \qquad U'_1W_1 = 0, \qquad U_1W'_1 = 0.$$

If $\mathfrak{V}$ is the linear subspace spanned by the vectors U_1, $\mathfrak{V}_y$ the totally orthogonal subspace,

$$V' \varepsilon \mathfrak{V}_y, \qquad W'_1 \varepsilon \mathfrak{V}_1, \qquad V_1 = V_1^y + V_1^0, \qquad W_1 = W_1^y + W_1^0,$$

$$U_1 \varepsilon \mathfrak{V}_x, \qquad V_1^0 = \text{const} \varepsilon \mathfrak{V}_x \qquad W_1^0 = \text{const} \varepsilon \mathfrak{V}_y, \qquad V_1^y \varepsilon \mathfrak{V}^y, \qquad W_1^y \varepsilon \mathfrak{V}^y.$$

We must have also $V_1^0U'_1 = W_1^0U'_1 = 0$. If $\mathfrak{V}_{x_1}$ is the plane spanned by the two constant vectors V_1^0, W_1^0 and $\mathfrak{V}_{x_2}$ the totally orthogonal subspace in $\mathfrak{V}_x$, $U'_1 \varepsilon \mathfrak{V}_{x_2}$. We obtain therefore the following decomposition of U, V, W:

in $\mathfrak{V}_{x_1}$:	U_1^0	V_1^0	W_1^0
in $\mathfrak{V}_{x_2}$:	$\bar{U}_1$	0	0
in $\mathfrak{V}_y$:	0	V_1^y	W_1^y

with $U_1^0 \cdot V_1^0 = -U_1^0 \cdot W_1^0 = 1$.

We omit the integration constant for V and W (it is equivalent to a translation), and obtain the following decompositions of U, V, W:

in $\mathfrak{V}_{x_1}$:	$U_1^0\Psi(u) + U$	$-V_1^0/2v$	$-W_1^0/2w$
in $\mathfrak{V}_{x_2}$:	$\bar{U}$	0	0
in $\mathfrak{V}_y$:	U_y^0	V_y	W_y.

The isotropy condition $(\partial_v M)^2 = 0$ is: $(U^2/v^2) - V'^2 - 2(U \cdot V'/v^2) = 0$ which gives a quadratic equation in $\Psi(u)$, with coefficients depending only on v. Then $\Psi(u)$ is an absolute constant a, and U^1 may be absorbed in U_1^0. The situation is the same in $\mathfrak{V}_y$ and $\mathfrak{V}_{x_1}$, and, with an obvious change of notation, we obtain the following decompositions of U, V, W:

in $\mathfrak{V}_x$:	U_x	0	0
in $\mathfrak{V}_y$:	U_y^0	V_y	W_y.

With the additional scalar conditions:

$$U^2 = U_x^2 + (U_y^0)^2 = a^2 \qquad U'^2 = 1$$

$$V'^2_y - (2/v^2)\,V'_y \cdot U_y^0 + a^2/v^4 = 0$$

$$W'^2_y - (2/w^2)\,W'_y \cdot U_y^0 + a^2/v^4 = 0$$

all our conditions are satisfied, and the parametric representations of the two 3-manifolds are

$$M = (w - v/vw)U + V + W;$$

$$P = (w - v) \cdot U + \int v^2 V' dv + \int w^2 W' dw.$$

In order to obtain real manifolds we must have A and θ real. Since $v = \sqrt{A}\, e^{i\theta/2}$, $w = \sqrt{A}\, e^{-i\theta/2}$, v and w must be complex conjugate for a real point. Hence we must take for V and W the same analytic vector function $\mathfrak{F}$, having complex conjugate values for two complex conjugate values of the variable (i. e. the Maclaurin's expansion of $\mathfrak{F}(z)$ has all its coefficients real), and for $U(u)$ a purely imaginary vector $iU_r(u)$. Then

$$M = 2 \sin(\theta/2)/A \cdot U_r(u) + \mathfrak{F}(\sqrt{A}\, e^{i\theta/2}) + \mathfrak{F}(\sqrt{A}\, e^{-i\theta/2}).$$

Non-holonomic case.

We may suppose that $ds_M^2 = (du^2)^2 + 2(du^i)(du^{i^*})$. Let us recall that

$$\Gamma_{\rho\sigma}^{\tau} = \{\rho^\tau\sigma\} - \tfrac{1}{2}(\Omega^\tau{}_{\rho,\sigma} + \Omega^\tau{}_{\sigma,\rho} + \Omega_{\rho\sigma^\tau}), \Omega_{\rho\sigma^\tau} = \Omega_\rho{}^\tau{}_\sigma.$$

Then the conditions II a) and II b) may be written:

$$\Omega^{i^*}{}_{ia} = \Omega^i{}_{i^*a} = 0$$

$$\Omega^{i^*}{}_{ai^*} + \Omega^i{}_{ai} = i \tan\theta/2 \cdot \Omega^a{}_{ii^*}$$

$$\partial_i(\log A) = (1 - e^{i\theta})\Omega^a{}_{ai}$$

$$\partial_{i^*}(\log A) = (1 - e^{i\theta})\Omega^a{}_{ai^*}$$

$$\partial_{i^*}(Ae^{i\theta}) = \partial_{i^*}(Ae^{-i\theta}) = 0.$$

$$\partial_a(\log(Ae^{i\theta})) = e^{i\theta} \cdot \tan\theta/2 \cdot \Omega^a{}_{i^*i}$$

$$\partial_a(\log(Ae^{-i\theta})) = e^{-i\theta} \cdot \tan\theta/2 \cdot \Omega^a{}_{i^*i}$$

The conditions $\Omega^{i^*}{}_{ia} = \Omega^i{}_{i^*a} = 0$ suggest computing non-symmetrically with respect to i and i^*. The system $\{\partial_a f = 0, \partial_i f = 0\}$ is complete; we call w a first integral of it, and u_1 and v_1 two independent first integrals of the equation $\partial_{i^*} f = 0$. Now $\partial_a = a\partial_{u_1} + b\partial_{v_1}$, $\partial_i = a'\partial_{u_1} + b'\partial_{v_1}$, $\partial_{i^*} = c\partial_w$. But, since $\Omega^i{}_{ai^*} = 0$, $\partial_{i^*a} - \partial_{ai^*}$ must be a linear combination of ∂_a and ∂_{i^*}. It is $(a \cdot \partial_{u_1} c + b \cdot \partial_{v_1} c)\partial_w - c(\partial_w a \cdot \partial_{u_1} + \partial_{u_1} b \cdot \partial_{v_1})$. One must have $\partial_w a/a = \partial_w b/b$, or $a = b\phi(u_1, v_1)$. Now $\partial_a = b(\partial_{v_1} + \phi(u_1, v_1)\partial_{u_1})$. Let v be a first integral of $\partial_{v_1} f + \phi(u_1, v_1)\partial_{u_1} f = 0$, and u a function of u_1 and v_1 independent of v. Then $\partial_a = a_1\partial_u$, $\partial_i = b_1\partial_u + c_1\partial_v$, $\partial_{i^*} = c\partial_w$. The conditions $\partial_a M \cdot \partial_i M$

$= \partial_a M \cdot \partial_{i^*} M = (\partial_i M)^2 = (\partial_{i^*} M)^2 = 0$ give $(\partial_w M)^2 = \partial_u M \cdot \partial_w M = 0$. Thus, for the manifold (M), $ds^2 = Edu^2 + 2Fdudv + Gdv^2 + 2Hdvdw$, with $b_1/c_1 = -F/E$, $G = F/E^2$. $ds^2 = (Edu + Fdv)^2/E + 2Hdvdw$, and $\partial_i = c(\partial_v - (F/E)\partial_u)$. The fundamental system becomes, in the holonomic parameters u, v, w:

$$\partial_u P = A \cdot \partial_u M.$$

$$\partial_v P = A \cdot (F/E) \cdot (1 - e^{i\theta}) \cdot \partial_u M + A \cdot e^{i\theta} \cdot \partial_v M.$$

$$\partial_w P = A \cdot e^{-i\theta} \cdot \partial_w M.$$

$\partial_{i^*}(Ae^{i\theta}) = 0$ gives $\partial_w(Ae^{i\theta}) = 0$, or $A = e^{-i\theta} \cdot a(u, v)$. $\partial_i(Ae^{-i\theta}) = 0$ gives $E \cdot \partial_v(Ae^{-i\theta}) - F \cdot \partial_u(Ae^{-i\theta}) = 0$, or $E \cdot \partial_v(ae^{-2i\theta}) - F \cdot \partial_u(ae^{-2i\theta}) = 0$. (2)

Let us write the compatibility conditions:

a) (u, w) $A(1 - e^{-i\theta})\partial_{uw} M = \partial_u(Ae^{-i\theta})\partial_w M - \partial_w A \partial_u M.$

b) $(v, w)(AF/E)(1 - e^{i\theta})\partial_{uw} M + A(e^{i\theta} - e^{-i\theta})\partial_{vw} M$
$+ \partial_w((AF/E)(1 - e^{-i\theta}))\partial_u M - \partial_v(Ae^{-i\theta})\partial_w M = 0$

c) $(u, v)(AF/E)(1 - e^{i\theta})\partial_{u^2} M + A(e^{i\theta} - 1)\partial_{uv} M + Q\partial_u M + \partial_u(Ae^{i\theta})\partial_v M$
$= 0$

where

$$Q = \partial_u((AF/E)(1 - e^{i\theta})) - \partial_v A.$$

By scalar multiplication with $\partial_u M$ (1), $\partial_v M$ (2), $\partial_w M$ (3), we obtain:

a) 1. $A(1 - e^{-i\theta})\partial_w E = -2 \cdot \partial_w A \cdot E.$

2. $A(1 - e^{-i\theta})(\partial_w F + \partial_u H) = 2H\partial_u(Ae^{-i\theta}) - 2F\partial_w A.$

3. already verified.

b) 1. $(AF/2E)(1 - e^{i\theta})\partial_w E + E\partial_w((AF/E)(1 - e^{i\theta}))$
$+ (A/2)(e^{i\theta} - e^{-i\theta})(\partial_w F - \partial_u H) = 0$

2. $(AF/2E)(1 - e^{i\theta})(\partial_w F + \partial_u H) + F\partial_w((AF/E)(1 - e^{i\theta}))$
$+ (A/2)(e^{i\theta} - e^{-i\theta})\partial_w(F^2/E) - H\partial_v(Ae^{-i\theta}) = 0$

3. already verified.

c) 1. $(AF/2E)(1 - e^{i\theta})\partial_u E + (A/2)(-1 + e^{i\theta})\partial_v E + QE$
$+ (F^2/E)\partial_u(Ae^{i\theta}) = 0.$

$$2.\ (AF/2E)\,(1-e^{i\theta})\,(2\partial_u F-\partial_v E)+\tfrac{1}{2}A(e^{i\theta}-1)\partial_u(F^2/E)$$
$$+QF+(F^2/E)\partial_u(Ae^{i\theta})=0.$$

$$3.\ (AF/2E)\,(1-e^{i\theta})\partial_w E+A/2(1-e^{i\theta})\,(\partial_u H-\partial_w F)$$
$$-H\partial_u(Ae^{i\theta})=0.$$

1) Taking $A=a(u,v)e^{-i\theta}$ and integrating a) 1 with respect to w, we obtain: $\sqrt{E}=b(u,v)\cdot(1-e^{i\theta})$.

2) Using these values of A and E, b) 1 and c) 3 may be written

$$(e^{i\theta}+1)\partial_u H=(1-e^{i\theta})\partial_w F+2iFe^{i\theta}\partial_w\theta=R$$

and

$$(e^{i\theta}+1)H\partial_u a/a=-[(1-e^{i\theta})\partial_w F+2iFe^{i\theta}\partial_w\theta]=-R.$$

If $\partial_u a=0$, then $R=0$,

$$F=c(u,v)\,(1-e^{i\theta})^2 \text{ and } Edu+Fdv=D(u,v,w)du_1$$

u_1 being a suitable function of u and v. Replacing u by u_1, we get $F=0$, and we are actually in the holonomic case. If $a\neq 0$, we may write

$$a\partial_u H+H\cdot\partial_u a=0, \text{ or } H=\phi(v,w)/a(u,v).$$

3) Substituting in a) 2 the values of H and $\partial_u H$ obtained above, and factoring out R, we obtain ($R\neq 0$ unless we are in the holonomic case)

$$e^{-2i\theta}\cdot\partial_u a-\partial_u(ae^{-2i\theta})=0.$$

Therefore, $\partial_u\theta=0$, $\theta=\theta(v,w)$.

4) c) 2 is the same as c) 1 multiplied by F/E.

5) Since $\partial_u\theta=0$, (2) can be written $\partial_v(ae^{-2i\theta})-F/E\cdot e^{2i\theta}\cdot\partial_u a=0$. And c) 1 becomes $\partial_u(aFe^{-i\theta}/b(1-e^{i\theta})^2)=\partial_v(abe^{-i\theta})$ with the solution $abe^{-i\theta}=\partial_u\Phi$, $aFe^{-i\theta}/b(1-e^{i\theta})^2=\partial_v\Phi$. Therefore $\partial_u(\Phi\cdot e^{i\theta})$ does not depend on w and may be written $\partial_u c(u,v)$. Hence

$$\Phi=c(u,v)e^{-i\theta}+\lambda_1(v,w),\qquad F/E=\partial_v\Phi/abe^{i\theta}=\partial_v\Phi/\partial_u\Phi$$

(since $ab=\partial_u c$, and $E=b^2(1-e^{i\theta})^2$). Since $F/E=\partial_v(ae^{-2i\theta})/\partial_u(ae^{-2i\theta})$, the jacobian $D(\Phi,ae^{-2i\theta})/D(u,v)=0$, and $\Phi=\Phi(ae^{-2i\theta},w)$ (3). But $\Phi=c(u,v)e^{-i\theta}+\lambda_1(v,w)$ (4). Φ must depend on u (unless $ab=0$ which is impossible). From its form (3) it depends on u through $ae^{-2i\theta}$, and from its form (4) only through the first term. Hence $c(u,v)e^{-i\theta}=\Psi(ae^{-2i\theta})$. If θ does not depend on w, neither does F/E, and we are in the holonomic

case. If θ depends on w, we must take $c = \sqrt{a}$, $\Psi = \sqrt{ae^{-2i\theta}}$, and $\Phi = \sqrt{a}e^{-i\theta} + W(w)$. Then $b = \partial_u a/(2a\sqrt{a})$. We may now suppose that $a = u$. This change of variable does not change the forms of the ds^2 and of the fundamental system. Then $b = 1/(2u\sqrt{u})$, $E = 1/4u^3(1 - e^{i\theta})$, $F/E = -2iu\partial_v\theta$, $F = i/2u^2(1 - e^{i\theta})\partial_v\theta$.

6) Substituting these values in c) 3 we obtain

$$Hu = (i(1 - e^{i\theta})^3/2(1 + e^{i\theta}))\partial_{vw}\theta$$

which is compatible with the results of 2).

7) With these values b) 2 is identically verified.

Thus all the projections of the compatibility conditions on the tangent 3-plane are verified; four of our nine conditions were consequences of the five others since we had used four conditions in the "study with α, i, i^*."

With $\lambda(v, w) = e^{i\theta}$, the fundamental system is:

$$\partial_u P = u \cdot \lambda \cdot \partial_u M.$$

$$\partial_v P = u \cdot \partial_v M + 2u^2(1 - 1/\lambda) \cdot \partial_v\lambda \cdot \partial_u M.$$

$$\partial_w P = u \cdot \lambda^2 \cdot \partial_w M.$$

The compatibility conditions are now

$$u\lambda(\lambda - 1)\partial_{uw}M = u \cdot \lambda'_w \cdot \partial_u M - \lambda^2 \cdot \partial_w M.$$

$$u(1-\lambda^2) \cdot \partial_{uw}M = 2u^2(1 - 1/\lambda)\lambda'_v \cdot \partial_{uw}M = 2u \cdot \lambda \cdot \lambda'_v \cdot \partial_w M$$
$$- 2u^2 \cdot \partial_w((1 - 1/\lambda)\lambda'_v) \cdot \partial_u M.$$

$$u(\lambda - 1) \cdot \partial_{uv}M - 2u^2(1 - 1/\lambda) \cdot \lambda'_v \cdot \partial_u{}^2 M = \partial_v M + u \cdot \lambda'_v(3 - 4/\lambda)\partial_u M.$$

By scalar multiplication with the normal vectors n_a, we obtain the following necessary and sufficient conditions for the coefficients of the second forms

$$a_{Muw,a} = a_{Mvw,a} = 0, \qquad \lambda a_{Muv,a} = 2u \cdot \lambda'_v \cdot a_{Muu,a}.$$

And the ds^2 must be

$$ds^2{}_M = \frac{(1-\lambda)^2}{u}\left[\left(\frac{du}{2u} - \frac{\lambda' v}{\lambda}\,dv\right)^2 + \frac{1-\lambda}{1+\lambda}\left(\frac{\lambda'' vw}{\lambda} - \frac{\lambda' v\lambda' w}{\lambda^2}\right)dvdw\right].$$

Using θ the necessary and sufficient conditions are

$$ds^2{}_M = \frac{(1 - e^{i\theta})^2}{u}\left[\left(\frac{du}{2u} - i\theta'_v dv\right)^2 + i \cdot \tan\frac{\theta}{2} \cdot e^{-i\theta} \cdot \theta''_{vw} \cdot dvdw\right].$$

$$a_{Muw,a} = a_{Mvw,a} = 0, \qquad a_{Muv,a} = 2iu \cdot \theta'_v \cdot a_{Muu,a}.$$

VIII. General Holonomic Case.

We consider the fundamental system

$$\partial_\alpha P = A \cdot \partial_\alpha M, \quad \partial_\lambda P = -A \cdot \partial_\lambda M, \quad \partial_k P = Ae^{i\theta_k} \cdot \partial_k M, \quad \partial_{k^*} P = Ae^{-i\theta_k} \partial_{k^*} M$$

with $ds^2_M = g_{\alpha\alpha}(du^2)^2 + \cdots + g_{\lambda\lambda}(du^\lambda)^2 + \cdots + 2g_{kk^*}du^k \cdot du^{k^*} + \cdots$

the coordinate directions being holonomic.

Compatibility conditions.

a) In k, k^*: $Ai \cdot \sin\theta_k \cdot \partial_{kk^*}M = \partial_k(Ae^{-i\theta_k}) \cdot \partial_{k^*}M - \partial_{k^*}(Ae^{i\theta_k}) \cdot \partial_k M.$

By scalar multiplication with the tangent vectors we obtain:

$$\partial_\alpha g_{kk^*} = \partial_\lambda g_{kk^*} = \partial_j g_{kk^*} = \partial_k(Ae^{-i\theta_k}) = \partial_k(Ae^{i\theta_k}) = 0.$$

b) In j, k: $A(e^{i\theta_j} - e^{i\theta_k}) \cdot \partial_{jk}M = \partial_j(Ae^{i\theta_k}) \cdot \partial_k M - \partial_k(Ae^{i\theta_j}) \cdot \partial_j M.$

By $\partial_\alpha M, \partial_\lambda M, \partial_\rho M, \partial_j M$: identically verified.

By $\partial_{j^*}M$: $A/2(e^{i\theta_j} - e^{i\theta_k}) \cdot \partial_k g_{jj^*} = -\partial_k(Ae^{i\theta_j})$.

But, because of a), $\partial_k(Ae^{i\theta_j}) = 0$, and similarly $\partial_k(Ae^{-i\theta_j}) = 0$.

c) In α, k: $A(1 - e^{i\theta_k}) \cdot \partial_{\alpha k}M = \partial_k M - \partial_\alpha(Ae^{i\theta_k}) \cdot \partial_k M.$

By $\partial_\beta M, \partial_\lambda M, \partial_j M, \partial_k M$: identically verified.

By $\partial_\alpha M$: $\frac{1}{2}(\partial_k g_{\alpha\alpha}/g_{\alpha\alpha}) = \partial_k A/A(1 - e^{i\theta_k})$.

By $\partial_{k^*}M$: $\partial_\alpha(Ae^{i\theta_k}) = 0$ (since $\partial_\alpha g_{kk^*} = 0$).

d) We obtain only, in λ, k:

$$\tfrac{1}{2}(\partial_k g_{\lambda\lambda}/g_{\lambda\lambda}) = \partial_k A/A(1 - e^{i\theta_k}) \text{ and } \partial_\lambda(Ae^{i\theta_k}) = 0.$$

e) In α, β: $\partial_\alpha A \cdot \partial_\beta M - \partial_\beta A \cdot \partial_\alpha M = 0.$

Therefore $\partial_\alpha A = \partial_\beta A = 0$, if there exists at least two positive directions. In the same way $\partial_\lambda A = \partial_\mu A = 0$, if there exists at least two negative directions.

f) In α, λ: $2A \cdot \partial_{\alpha\lambda}M = \partial_\alpha A \cdot \partial_\lambda M - \partial_\lambda A \cdot \partial_\alpha M.$

We see that $Ae^{i\theta_k}$ depends only on u^k, $Ae^{-i\theta_k}$ only on u^{k^*}. If they are not constant we can write $Ae^{i\theta_k} = (u^k)^2$, $Ae^{-i\theta_k} = (u^{k^*})^2$. Then $A = u^k u^{k^*}$. And since it is impossible to have also $A = u^j u^{j^*}$, $Ae^{i\theta_j}$ and $Ae^{-i\theta_j}$ must be constant. Then A is constant which is contrary to $A = u^k u^{k^*}$. Hence there is only one couple of isotropic coordinate directions.

Case with one couple of isotropic principal directions.

Let $u^k = v$, $u^{k^*} = w$. Then $A = vw$, and $\partial_\alpha A = \partial_\lambda A = 0$ are verified. $e^{i\theta} = v/w$, $e^{-i\theta} = w/v$.

Consider the compatibility conditions

$$\partial_{vw} M = 0, \quad \partial_{v\alpha} M = -(w/v(w-v))\partial_\alpha M, \quad \partial_{w\alpha} M = (v/w(w-v)) \cdot \partial_\alpha M,$$

and the similar ones with λ instead of α. Their integration gives

$$M = ((w-v)/vw) U(u^\alpha, \cdots, u^\lambda, \cdots) + V(v) + W(w).$$

The ds^2-conditions impose that:

1) U satisfies the same conditions as M in VI (real principal directions). Only the two first cases can happen; V case 3) cannot happen since $\partial_\alpha A = \partial_\lambda A = 0$.

2) There exist two supplementary subspaces $\mathfrak{B}_x$ and $\mathfrak{B}_y$ with the following decompositions of U, V, W (see: VII, holonomic case):

In $\mathfrak{B}_x$:	U_x	0	0.
In $\mathfrak{B}_y$:	$U^0{}_y$	(const.) V	W

3) $U^2 = a^2 =$ constant,

$$V'^2 - (2/v^2) V' \cdot U^0{}_y + (a^2/v^4) = 0, \quad W'^2 - (2/v^2) \cdot W' \cdot U^0{}_y + (a^2/w^4) = 0.$$

Several couples of isotropic principal directions.

Then all the angles of rotation θ_κ are constant, and, since $\partial_\kappa (Ae^{i\theta_j}) = 0$, A depends only on the parameters corresponding to the real principal directions. But $\partial_\alpha (Ae^{i\theta_\kappa}) = \partial_\lambda (Ae^{i\theta_\kappa}) = 0$. Therefore A is also a constant. After a homothety we can take $A = 1$, and the manifolds are *isometric*.

The compatibility conditions are $\partial_{\alpha\lambda} M = 0$, $\partial_{\alpha j} M = \partial_{\lambda j} M = 0$ (because $\theta_j \neq k\pi$; in the case $\theta_j = k\pi$, these directions are real). And $\partial_{jj^*} M = 0$, $\partial_{jk} M = 0$ (if $\theta_j \neq \theta_k$).

Denoting by $j_1, j_2, \cdots$ the indices corresponding to the same angle we get the following parametric representation:

$$M = M_1(u^\alpha, \cdots) + M_2(u^\lambda, \cdots) + M_j(u^{j_1}, \cdots, u^{j_n}) + M^*_j(u^{j^*_1}, \cdots, u^{j^*_n}) + \cdots + M_k(u^{k_1}, \cdots, u^{k_p}) + M^*_k(u^{k^*_1}, \cdots, u^{k^*_p})$$

with the following conditions due to the ds^2. We do not consider the conditions $g_{j_i j^*_2} = 0$ which are not necessary (see III, 4).

a) $M_1, M_2, M_j, M^*_j, \cdots, M_k$ and M^*_k vary in mutually orthogonal linear subspaces.

b) The manifolds (M_j) and $(M^*_j), \cdots, (M_k)$ and (M^*_k), are totally isotropic, and complex conjugate in the real case.

IX. Isometric Manifolds.

We shall now study the case where the manifolds (M) and (P) are not only in conformal, but in isometric correspondence.

Preliminary formula: principal normal to a curve on a manifold.

We consider the manifold $M(u^\alpha, \cdots)$ in the euclidean N-space, and on it the curve C, $u^\alpha = u^\alpha(s)$, s being the arc on C. The primes denote the derivations with respect to s: $g_{\alpha\beta} \cdot u'^\alpha \cdot u'^\beta = 1$ by definition of the arc. The tangent unit vector of C is $\vec{t} = dM/ds = \partial_\alpha M \cdot u'_\alpha$. If $\vec{n}$ denotes the principal normal unit vector, and R the first radius of curvature of C, we have by definition $d^2M/ds^2 = \vec{n}/R$, and, by the Frenet formulae of the manifold

$$\vec{n}/R = d/ds(\partial_\alpha M \cdot u'^\alpha) = \partial_\alpha M \cdot u''^\alpha + \partial_{\alpha\beta} M \cdot u'^\alpha \cdot u'^\beta = (u''^\lambda + \Gamma^\lambda_{\alpha\beta} u'^\alpha \cdot u'^\beta) \cdot \partial_\lambda M + a^i_{\alpha\beta} \cdot u'^\alpha \cdot u'^\beta \cdot \vec{n_i}.$$

If $\vec{n}$ is tangent to the manifold, $a^i_{\alpha\beta} \cdot u'^\alpha \cdot u'^\beta = 0$ (asymptotic curves). If $\vec{n}$ is normal, $u''^\lambda + \Gamma^\lambda_{\alpha\beta} \cdot u'^\alpha \cdot u'^\beta = 0$ (geodesics).

A property of the geodesics.

Since here the two manifolds are isometric, the geodesics are corresponding curves. We consider two corresponding geodesics, with the same arc s, and with tangent unit vectors $\vec{t}(s)$, $\vec{t'}(s)$, with angle V between them; $\cos V = \vec{t} \cdot \vec{t'}$.

$\sin V \cdot dV/ds = \vec{t} \cdot \vec{n}'/R' + \vec{t}' \cdot \vec{n}/R = 0$, since $\vec{n}$ and $\vec{n}'$ are normal vectors. *Hence the angle of rotation, which is applied to a tangent vector t on (M) in order to obtain the corresponding tangent vector on (P), is stationary for a motion of M in the direction of* $\vec{t}$.

If the manifolds are surfaces, and if the correspondence between tangent planes is a proper rotation, the angle of rotation is the same in all directions, and is evidently constant on every part of the surface whose two points can be joined by geodesics.

Constancy of the rotation angles.

This fact may be generalized. We shall prove that, when the rotation is canonically decomposed into rotations in mutually orthogonal planes, with angles θ_κ, *these angles are constant.* We denote by $b^\alpha{}_\lambda$ the linear operator of rotation, by p^α the contravariant components of a tangent vector. By rotation it is transformed to $q^\alpha = b^\alpha{}_\lambda \cdot p^\lambda$. The angle V between these vectors satisfies $\cos V = g_{\alpha\beta} p^\beta q^\alpha = g_{\alpha\beta} b^\alpha{}_\lambda p^\lambda p^\beta$. By parallel displacement along the geodesic tangent to p^α at M, the covariant derivative $p^\mu \nabla_\mu p^\alpha$ is zero. The covariant derivative of the scalar $\cos V$ is the ordinary derivative which is also zero as proved before. Hence $g_{\alpha\beta} p^\beta p^\lambda p^\mu \nabla_\mu (b^\alpha{}_\lambda) = 0$. This relation is true for every vector p^α; being a polynomial in the p^α's, its coefficients must be all zero; $g_{\alpha(\beta} \nabla_\mu (b^\alpha{}_{\lambda)}) = 0$. (The parenthesis around the indices β,μ,λ denotes, as usual, that one has to take the sum of the 6 terms corresponding to all the permutations of the 3 indices β,μ,λ.)

If, at the point M, we take geodesic coordinates with the principal directions as coordinate directions, the first covariant derivatives are, at M, equal to the ordinary derivatives. Hence, using indices $a, b, c \cdots$, in order not to interfere with the conventions of III, $g_{a(b} \nabla_d (b^\alpha{}_{c)}) = 0$ can be written

$$g_{ab}\partial_d s_a + g_{ba}\partial_d s_b + g_{da}\partial_b s_d + g_{ad}\partial_b s_a + g_{bd}\partial_a s_b + g_{db}\partial_a s_d = 0$$

or

$$g_{ab}\partial_d(s_a + s_b) + g_{ad}\partial_b(s_a + s_d) + g_{bd}\partial_a(s_b + s_d) = 0. \quad \text{(E)}$$

But here ${}_a s = 1$, ${}_\lambda s = -1$, ${}_k s = e^{\theta i_k}$, ${}_{k^*} s = e^{-i\theta_k}$, $g_{aa} = g_{\lambda\lambda} = g_{kk^*} = 1$ the other g_{ab} being zero. With two Greek indices, or with two indices j, k, (E) is verified. With k, k^*, and any other index a: $g_{kk^*}\partial_a(e^{i\theta_k} + e^{-i\theta_k}) = 0$, $\partial_a \theta_k = 0$. With k, k, k^* : $g_{kk^*}\partial_k(e^{i\theta_k} + e^{-i\theta_k}) + g_{kk^*} \cdot \partial_k(e^{i\theta_k} + e^{-i\theta_k}) = 0$.

Hence $\partial_{\kappa^*}\theta_\kappa = 0$, and similarly $\partial_{\kappa^*}\theta_\kappa = 0$. Therefore the θ_κ are constant. This result is also true if some θ's are equal.

We shall now show that, in the 3- and 4-dimensional cases, the principal directions are holonomic, so that we are concerned with a case already studied in VIII.

We shall take, as in the 3-dimensional non-holonomic case,

$$ds^2_M = (du^a)^2 + \cdots + (du^\lambda)^2 + \cdots + 2(du^\kappa)(du^{\kappa^*}) + \cdots,$$

and write the compatibility conditions in terms of the Ω's. We shall not use the constancy of the θ_κ (the computation is only slightly simplified by using this fact) but prove it a second time.

3-dimensional case.

Putting $A = 1$ in the conditions (1) of VII, non-holonomic case, we get

$$\Omega^a{}_{ai} = \Omega^a{}_{ai^*} = \Omega^{i^*}{}_{ia} = \Omega^i{}_{i^*a} = \partial_i\theta = \partial_{i^*}\theta = 0.$$

$$\partial_a(\log(Ae^{i\theta})) = i\partial_a\theta = e^{i\theta}\tan(\theta/2)\,\Omega^a{}_{i^*i}.$$

$$\partial_a(\log(Ae^{i\theta})) = -i\partial_a\theta = e^{-i\theta}\tan(\theta/2)\,\Omega^a{}_{i^*i}.$$

Since $\theta \neq \kappa\pi$, $\partial_a\theta = 0$, $\Omega^a{}_{i^*i} = 0$. Hence θ *is a constant,* and the conditions for the Ω's prove immediately that there exists 3 parameters u, v, w such that

$$\partial_a = a_1\partial_u, \qquad \partial_i = a_2\partial_v, \qquad \partial_{i^*} = a_3\partial_w.$$

Therefore the principal directions are holonomic.

Compatibility conditions in the general case.

We do not write the intermediary computations which are long but without mathematical difficulties.

1) In α, β.

$\alpha, \beta, \gamma,$: verified

λ : $\Omega^\lambda{}_{a\beta} = 0$

k : $(e^{i\theta_\kappa} - 1)\Omega^\kappa{}_{a\beta} = 0$. Hence $\Omega^\kappa{}_{a\beta} = 0$.

2) In λ, μ :

α : $\Omega^a{}_{\lambda\mu} = 0$.

λ, μ, ν : verified.

k : $\Omega^{k}{}_{\lambda\mu} = 0$.

3) In α, λ :

α : $\Omega^{(\alpha)}{}_{(\alpha)\lambda} = 0$.

β : $\Omega^{\beta}{}_{a\lambda} + \Omega^{a\beta}{}_{\beta\lambda} = 0$ (using 1), λ).

λ : $\Omega^{(\lambda)}{}_{a(\lambda)} = 0$.

μ : $\Omega^{\mu}{}_{a\lambda} + \Omega^{\lambda}{}_{a\mu} = 0$ (using 2), α).

k : $e^{i\theta_k}\Omega^{k}{}_{a\lambda} + \Omega^{\lambda}{}_{k^*a} + \Omega^{a}{}_{k^*\lambda} = 0$.

k^* : $e^{-i\theta_k}\Omega^{k^*}{}_{a\lambda} + \Omega^{\lambda}{}_{ka} + \Omega^{a}{}_{k\lambda} = 0$.

4) In α, k (or λ, k).

α : $\Omega^{(\alpha)}{}_{(\alpha)k} = 0$, and $\Omega^{(\lambda)}{}_{(\lambda)k} = 0$.

β : $\Omega^{\beta}{}_{ak} + \Omega^{a}{}_{\beta k} = 0$, and $\Omega^{\mu}{}_{\lambda k} + \Omega^{\lambda}{}_{\mu k} = 0$.

λ : $\Omega^{\lambda}{}_{ka} = 0$, and $\Omega^{a}{}_{k\lambda} = 0$.

Hence, using 3), k^* : $\Omega^{k^*}{}_{a\lambda} = \Omega^{k}{}_{a\lambda} = 0$.

k : $\Omega^{a}{}_{k^*k} + \Omega^{(k^*)}{}_{(k^*)a} + \Omega^{(k)}{}_{(k)a} = 2(\partial_a(e^{i\theta_k})/e^{i\theta_k} - 1)$.

Similarly: $\Omega^{a}{}_{kk^*} + \Omega^{(k)}{}_{(k)a} + \Omega^{(k^*)}{}_{(k^*)a} = -2(\partial_a(e^{-i\theta_k})/e^{-i\theta_k} - 1)$

and the same with λ, but with $e^{i\theta_k} + 1$ and $e^{-i\theta_k} - 1$ in denominators.

j : $(2e^{i\theta_j} - e^{i\theta_k} - 1)\Omega^{j}{}_{ak} = (e^{i\theta_k} - 1)(\Omega^{a}{}_{k^*k} + \Omega^{k}{}_{j^*a})$

and $(2e^{i\theta_j} - e^{i\theta_k} + 1)\Omega^{j}{}_{\lambda k} = (e^{i\theta_k} + 1)(\Omega^{\lambda}{}_{j^*k} + \Omega^{k}{}_{j^*\lambda})$.

5) In k, k^*.

α : $\Omega^{a}{}_{kk^*}(1 - \cos\theta_k) = i \sin\theta_k(\Omega^{(k)}{}_{(k)a} + \Omega^{(k^*)}{}_{(k^*)a})$.

Comparing with 4), k : $\Omega^{a}{}_{kk^*} = \Omega^{(k)}{}_{(k)a} = \Omega^{(k^*)}{}_{(k^*)a} = \partial_a\theta_k = 0$

λ : in the same way $\Omega^{\lambda}{}_{kk^*} = \Omega^{(k)}{}_{(k)\lambda} = \Omega^{(k^*)}{}_{(k^*)\lambda} = \partial_\lambda\theta_k = 0$.

k : $\partial_{k^*}\theta_k = 0$ (the terms in Ω disappear).

Similarly $\partial_k \theta_k = 0$.

$$j : \Omega^j{}_{kk^*}(e^{i\theta_j} - \cos\theta_k) + (\Omega^{(k)}{}_{j^*(k)} + \Omega^{(k^*)}{}_{j^*(k^*)})\, i \sin\theta_k = 0.$$

$$j^* : \Omega^{j^*}{}_{kk^*}(e^{-i\theta_j} - \cos\theta_k) + (\Omega^{(k)}{}_{j(k)} + \Omega^{(k^*)}{}_{j(k^*)})\, i \sin\theta_k = 0.$$

6) In k,j :

$$\alpha : \Omega^\alpha{}_{kj}(2 - e^{i\theta_k} - e^{i\theta_j}) = (e^{i\theta_j} - e^{i\theta_k})(\Omega^{j^*}{}_{\alpha k} + \Omega^{k^*}{}_{\alpha j}).$$

$$\lambda : \Omega^\lambda{}_{kj}(-2 - e^{i\theta_k} - e^{i\theta_j}) = (e^{i\theta_j} - e^{i\theta_k})(\Omega^{j^*}{}_{\lambda k} + \Omega^{k^*}{}_{\lambda j}).$$

$$k^* : (e^{i\theta_j} - e^{-i\theta_k})\Omega^{k^*}{}_{kj} = 0.$$

$$k : \Omega^{(k)}{}_{(k)j} + \Omega^{k^*}{}_{(k^*)j} + \Omega^{j^*}{}_{kk^*} = 2\partial_j(e^{-i\theta_k})/e^{i\theta_j} - e^{i\theta_k}.$$

Exchanging k and k^*,

$$\Omega^{(k)}{}_{(k)j} + \Omega^{(k^*)}{}_{(k^*)j} + \Omega^{j^*}{}_{kk^*} = 2\partial_j(e^{-i\theta_k})/e^{i\theta_j} - e^{i\theta_k}.$$

Comparing with 5), j,

$$\partial_j \theta_k = 0, \qquad \Omega^j{}_{kk^*} = \Omega^{(k)}{}_{(k)j} + \Omega^{(k^*)}{}_{(k^*)j} = 0$$

(even if $\theta_j = -\theta_k$).

$$1 : \Omega^l{}_{jk}(2e^{i\theta_l} - e^{i\theta_k} - e^{i\theta_j}) = (e^{i\theta_j} - e^{i\theta_k})(\Omega^{j^*}{}_{l^*k} + \Omega^{k^*}{}_{l^*j}).$$

We see that *the angles θ_k are constant.*

Holonomy in the 4-dimensional case.

We shall now prove that, in the 4-dimensional case, we have enough symbols Ω equal to zero in order to make the principal directions holonomic. In the general case, when the four characteristic roots are distinct, we notice that every system of two equations extracted from the system

$$\{\partial_\alpha f = \partial_\lambda f = \partial_k f = \partial_{k^*} f = 0\} \quad (\text{or } \{\partial_k f = \partial_{k^*} f = \partial_j f = \partial_{j^*} f = 0\})$$

is complete.

LEMMA. *If every system of two equations extracted from*

$$\{\partial_\alpha f = \partial_\lambda f = \partial_k f = \partial_{k^*} f = 0\},$$

is complete, then the non-holonomic derivatives $\partial_\alpha, \partial_\lambda, \partial_k, \partial_{k^*}$ are proportional to holonomic derivatives.

The system $\{\partial_\lambda f = \partial_k f = \partial_{k^*} f = 0\}$, being complete, has a first integral u. Let us denote by v, w, t three independent first integrals of $\partial_a f = 0$. Then

$$\partial_a = a\partial_u, \qquad \partial_\lambda = a_1\partial_v + b_1\partial_w + c_1\partial_t,\ \partial_k = a_2\partial_v + b_2\partial_w + c_2\partial_t,$$

$$\partial_{k^*} = a_3\partial_v + b_3\partial_w + c_3\partial_t.$$

$$\partial_{a\lambda} - \partial_{\lambda a} = a(\partial_u a_1 \cdot \partial_v + \partial_u b_1 \cdot \partial_w + \partial_u c_1 \cdot \partial_t) - (a_1 \cdot \partial_t a + b_1 \cdot a + c_1 \partial_t a \partial_u).$$

But, since $\Omega^k{}_{a\lambda} = \Omega^{k^*}{}_{a\lambda} = 0$, $\partial_{a\lambda} - \partial_{\lambda a}$ is a linear combination of ∂_a and ∂_λ. Hence we must have $\partial_u a_1/a_1 = \partial_u b_1/b_1 = \partial_u c_1/c_1$, whence

$$a_1 = \phi_1(u, v, w, t)a'_1(v, w, t), \quad b_1 = \phi_1(u, v, w, t)b'_1(v, w, t), \quad c_1 = \phi_1 c'_1$$

and similarly for $a_2, b_2, c_2, a_3, b_3, c_3$. Now

$$\partial_\lambda = \phi_1 \cdot (a'_1\partial_v + b'_1\partial_w + c'_1\partial_t), \qquad \partial_k = \phi_2 \cdot (a'_2\partial_v + b'_2\partial_w + c'_2\partial_t),$$
$$\partial_{k^*} = \phi_3 \cdot (a'_3\partial_v + b'_3\partial_w + c'_3\partial_t).$$

Hence the equations $\{\partial_\lambda f = \partial_k f = \partial_{k^*} f = 0\}$ do not contain the variable u. We make the same reasoning for them; we isolate ∂_λ, and, since $\Omega^\lambda{}_{kk^*} = 0$, we change the variables v, w, t into v', w', t' such that

$$\partial_\lambda = \phi_1 \cdot l \cdot \partial_{v'}, \qquad \partial_k = \phi_2 \cdot (l_1(v', w', t')\partial_{w'} + m_1(v', w', t')\partial_{t'}),$$
$$\partial_{k^*} = \phi_3 \cdot (l_2(v', w', t')\partial_{w'} + m_2(v', w', t')\partial_{t'}).$$

Since

$$\Omega^{k^*}{}_{k\lambda} = \Omega^k{}_{k^*j} = 0, \qquad \partial_{v'}(\phi_2 l_1)/\phi_2 l_1 = \partial_{v'}(\phi_2 m_1)/\phi_2 m_1$$

and

$$\partial_{v'}(\phi_3 l_2)/\phi_3 l_2 = \partial_{v'}(\phi_3 m_2)/\phi_3 m_2$$

or $\partial_{v'} l_1/l_1 = \partial_{v'} m_1/m_1$ and $\partial_{v'} l_2/l_2 = \partial_{v'} m_2/m_2$, which are solved by

$$l_1 = \psi_1(v', w', t') \cdot l'_1(w', t'), \qquad m_1 = \psi_1 \cdot m'_1(w', t')$$
$$l_2 = \psi_2(v', w', t') \cdot l'_2(w', t'), \qquad m_2 = \psi_2 \cdot m'_2(w', t').$$

Now $\partial_k = \phi_2 \cdot \psi_1 \cdot (l'_1(w', t')\partial_{w'} + m'_1 \cdot \partial_{t'})$, $\partial_{k^*} = \phi_3 \cdot \psi_2 \cdot (l'_2 \cdot \partial_{w'} + m'_2 \cdot \partial_{t'})$. Changing the variables w' and t' we can write

$$l'_1 \cdot \partial_{w'} + m'_1 \cdot \partial_{t'} = n_1(w', t')\partial_{w''},$$
$$l'_2 \cdot \partial_{w'} + m'_2 \cdot \partial_{t'} = n_2(w', t')\partial_{t''}.$$

Hence, in the 4-dimensional case (and more generally, in any dimension, in the case where there are only two couples of isotropic principal directions with different rotation angles, at most one positive and at most one negative principal direction) we are in the holonomic case.

If there are several principal directions corresponding to the same characteristic root, some conditions $\Omega = 0$ do not hold; but, by compensation, we have more freedom for choosing the principal directions, every direction of the linear space spanned by the principal directions corresponding to the same characteristic root being principal.

Therefore: *there exist principal directions which form a holonomic coordinate system, when there are at most two different rotation angles.*

Remark. This result is probably true in general. The conditions of the type 6), 1, with all the possible combinations of the 6 indices j, j^*, k, k^*, l, l^*, form a system of $6(6-1)/2 = 30$ linear homogeneous equations in 30 unknowns Ω. Their determinant is probably $\neq 0$ (if $\theta_j, -\theta_j, \theta_k, -\theta_k, \theta_l, -\theta_l$ are all different; otherwise it is already proved). Hence $\Omega^l{}_{jk} = 0, \cdots$, and the holonomy is proved by the lemma.

PRINCETON UNIVERSITY.

SUL CALCOLO DELLE SOLUZIONI DEI PROBLEMI AL CONTORNO PER LE EQUAZIONI LINEARI DEL SECONDO ORDINE DI TIPO ELLITTICO.*

By Luigi Amerio.

Introduzione.

Nei suoi *Appunti di Analisi Superiore* il Picone [1] tratta i classici problemi al contorno per le equazioni differenziali lineari a derivate parziali seguendo un unico criterio, basato su una nuova e suggestiva interpretazione, da Lui stesso indicata, della formula di Green.

In virtù di tale criterio, la soluzione dei problemi considerati si deduce immediatamente dalla preventiva risoluzione di alcuni problemi di Analisi che il Picone pone in modo ben preciso calendosi, se si considerano le equazioni lineari del secondo ordine, di tipo ellittico, delle seguenti considerazioni.

Sia

$$(1) \qquad E(u) = \Sigma a_{ik}(\partial^2 u/\partial x_i \partial x_k) + \Sigma b_i(\partial u/\partial x_i) + cu = f \qquad (a_{ik} = a_{ki})$$

un'equazione di tal natura, in m variabili $(x_1, \cdots, x_m)$, e si suppongano le a_{ik}, b_i funzioni reali di classe 2 e 1 rispettivamente,[2] le c, f reali e continue in un dominio limitato τ di S_m. Inoltre il determinate $\| a_{ik} \|$ sia unitario, ciò che manifestamente non lede la generalità.

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* Lavoro eseguito presso l'Istituto Nazionale per le Applicazioni del Calcolo. Received September 21, 1946.

[1] M. Picone, *Appunti di Analisi Superiore*, Rondinella, Napoli, 1940, pp. 752-65.

[2] Seguendo G. Ascoli, (*Equazioni alle derivate parziali dei tipi ellittico e parabolico*, Sansoni, Firenze, 1935, p. 53) diremo che una funzione u è di classe k in un dominio τ se è continua ensieme alle sue derivate parziali di ordine $\leq k$ in ogni punto interno a τ e se è possible prolungare la u e le sue derivate parziali di ordine $\leq k$ anche sul contorno σ di τ in modo da risultare continue in tutto τ. Diremo inoltre che la u è di classe kH in τ se le sue derivate parziali di ordine $\leq k$ soddisfano in tutto τ e una condizione di Hölder. Scriveremo H in luogo di oH.

Una varietà σ a $m-1$ dimensioni si dirà di classe k in un suo punto M se risulta definita, in un intorno completo di M da una rappresentazione (invertibile) delle coordinate dei suoi punti come funzioni di classe k di $m-1$ parametri variabili in un certo dominio, con matrice funzionale non nulla. Si dirà poi di classe k una varietà σ la quale sia di classe k in ogni suo punto.

Introdotta l'espressione differenziale aggiunta di $E(u)$:

$$(2)\quad E^*(w) = \Sigma(\partial^2(a_{ik}w)/\partial x_i \partial x_k) - \Sigma(\partial(b_i w)/\partial x_i) + cw$$
$$= \Sigma a_{ik}(\partial^2 w/\partial x_i \partial x_k) + \Sigma b^*{}_i(\partial w/\partial x_i) + c^* w,$$

supponiamo che il contorno σ di τ soddisfi a quelle condizioni che assicurano la validità della formula di Green per ogni coppia u, w di funzioni di classe 2 in τ:

$$-\int_\tau \{uE^*(w) - wE(u)\}d\tau = \int_\sigma \{u(\partial w/\partial \nu) - w(\partial u/\partial \nu) - Luw\}d\sigma,$$

dove ν è la direzione conormale,[3] orientata verso l'interno di τ, e si è posto

$$L = \Sigma \cos(nx_i)\{b_i - \Sigma(\partial a_{ik}/\partial x_k)\},$$

n esendo la normale a σ, orientata anch'essa verso l'interno di τ.

Risulta allora, se u è un integrale della (1) di classe 2 in τ,

$$(3)\quad -\int_\tau uE^*(w)d\tau = \int_\sigma \{u((\partial w/\partial \nu) - Lw) - (\partial u/\partial \nu)w\}d\sigma - \int_\tau fwd\tau$$

e da questa classica relazione il Picone deduce le seguenti proposizioni, che riferirmo, per semplicità, al problema di Dirichlet.

a) *Se si conosce una successione $\{v_r\}$ di integrali dell'equazione aggiunta*

$$(4)\qquad E^*(v) = 0$$

la quale sia chiusa,[4] su σ, della (4) si ricava il sistema di Fischer-Riesz:

$$(5)\qquad \int_\sigma \{u((\partial v_r/\partial \nu) - Lv_r) - (\partial u/\partial \nu)v_r\}d\sigma - \int_\tau fv_r d\tau = 0$$

e quindi si conoscono i coefficienti di Fourier dell'incognita $\partial u/\partial \nu$ rispetto alla sucessione $\{v_r\}$. La funzione $\partial u/\partial \nu$ resta perciò determinata nei punti di σ.

b) *Se si conosce una successione $\{w_r\}$ di funzioni di classe 2 in τ tali che la successione $\{E^*(w_r)\}$ sia chiusa, in τ, dalla (3) si ricava il sistema di Fischer-Riesz*:

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[3] ν è la retta di parametri direttori $\nu_i = \Sigma a_{ik}\cos(ux_k)$. Osserviamo che $\partial u/\partial \nu$ risulta *proporzionale* alla derivata della funzione u rispetto alla direzione ν: benchè impropriamente chiameremo, seguendo l'uso comune, tale funzione la *derivata conormale* di u.

[4] In questo enunciato, e nei seguenti, basterà considerare la chiusura rispetto alla totalità delle funzioni a quadrato sommabile nell'assegnato insieme di definizione. Noi però considereremo, in seguito, la chiusura anche rispetto alla totalità delle funzioni sommabili.

$$(6)\quad -\int_\tau uE^*(w_r)\,d\tau = \int_\sigma \{u((\partial w_r/\partial \nu) - Lw_r) - (\partial u/\partial \nu\, w_r\}d\sigma - \int_\tau f w_r d\tau$$

e, quindi, note, su σ, le u e $\partial u/\partial \nu$, resta determinata la u nei punti di τ.

Osserviamo che, mediante le successioni $\{v_r\}$, $\{w_r\}$ considerate in a), b), si ricava prima la $\partial u/\partial \nu$ su σ e successivamente la u in τ. È possibile però enunciare una terza proposizione relativa alla contemporanea determinazione di tali funzioni.

Cominciamo per questo, col ricordare la seguente estensione, indicata dal Picone,[5] della classica teoria degli sviluppi in serie di funzioni ortogonali e normali.

Siano $f_1, \cdots, f_p$, p funzioni reali definite rispettivamente nelle varietà $I_2, \cdots, I_p$ di S_m, aventi ciascuna un proprio numero $\leq m$ di dimensioni. Consideriamo tali funzioni come componenti di un vettore $\boldsymbol{F}$ dello spazio S_p. Se $\boldsymbol{\omega}$ è un altro vettore, di componenti $\omega_1, \cdots, \omega_p$ in $I_1, \cdots, I_p$ rispettivamente, chiameremo *prodotto integrale* di $\boldsymbol{F}$ per $\boldsymbol{\omega}$, e lo indicheremo col simbolo

$$(\boldsymbol{F}, \boldsymbol{\omega})$$

la somma

$$(\boldsymbol{F}, \boldsymbol{\omega}) = \sum_1^p{}_h \int_{I_h} f_h \omega_h dI_h.$$

Il prodotto integrale $(\boldsymbol{F}, \boldsymbol{F})$, si dirà la *norma integrale* di $\boldsymbol{F}$ e una successione $\{\boldsymbol{z}_r\}$ di vettori si dirà *ortogonale* e *normale* se risulterà

$$(\boldsymbol{z}_r, \boldsymbol{z}_s) = \begin{cases} 1 & \text{per } r = s \\ 0 & \text{per } r \neq s. \end{cases}$$

Da una qualsiasi successione $\{\boldsymbol{\omega}_r\}$ di vettori di norma integrale finita si può sempre, mediante l'estensione di un noto procedimento, dedurne una equivalente, $\{\boldsymbol{z}_r\}$, ortogonale e normale.

Consideriamo ora un sistema di equazioni di Fischer-Riesz:

$$(7)\quad (\boldsymbol{F}, \boldsymbol{\omega}_r) = c_r$$

dove $\boldsymbol{F}$ è un vettore incognito e le c_r sono costanti assegnate, che chiameremo i *coefficienti di Fourier* di $\boldsymbol{F}$ rispetto alla successione $\{\boldsymbol{\omega}_r\}$. Sostituito al sistema (7) l'equivalente sistema

$$(8)\quad (\boldsymbol{F}, \boldsymbol{z}_r) = d_r,$$

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[5] M. Picone, *loc. cit.*[1], pp. 632-34, Vedasi anche, dello stesso Autore, "Vedute unitarie sul calcolo delle soluzione delle equazioni alle derivate parziali della fisica matematica," *Atti I Convegno di Matematica Applicata*, Roma, 1936.

(dove le d_r risultano note, perchè ciascuna è combinazione lineare di un numero finito di c_r) vale il teorema di Fischer-Riesz: *condizione necessaria e sufficiente perchè il sistema* (7) *ammetta una soluzione* $\boldsymbol{F}$ *di norma integrale finita è che converga la serie* Σd^2_r. In tal caso la serie di vettori $\Sigma d_r \boldsymbol{z}_r$ converge in media a un vettore $\boldsymbol{F}'$ per il quale $(\boldsymbol{F}', \boldsymbol{\omega}_r) = c_r$, cioè, indicata con $z_{r,h}$ la componente di $\boldsymbol{z}_r$ in I_h, la serie $\Sigma d_r z_{r,h}$ converge in media nella varietà I_h alla componente f'_h del vettore $\boldsymbol{F}'$. Indicato poi con $\boldsymbol{F}''$ il più generale vettore, di norma integrale finita, ortogonale a tutti i vettori $\boldsymbol{\omega}_r$, cioè tale che risulti

$$(\boldsymbol{F}'', \boldsymbol{\omega}_r) = 0, \tag{9}$$

la soluzione $\boldsymbol{F}$ del sistema (7) (nella totalità dei vettori di norma integrale finita) è data dalla somma

$$\boldsymbol{F} = \boldsymbol{F}' + \boldsymbol{F}''. \tag{10}$$

Il vettore $\boldsymbol{F}''$ dipende linearmente da un certo numero di costanti arbitrarie. Inoltre, per una qualsiasi soluzione $\boldsymbol{F}$ del sistema (7) vale la disuguaglianza di Bessel: $\Sigma d^2_r \leq (\boldsymbol{F}, \boldsymbol{F})$. Infine condizione necessaria e sufficiente perchè la successione $\{\boldsymbol{\omega}_r\}$ sia *chiusa* rispetto alla totalità dei vettori di norma integrale finita, cioè il vettore $\boldsymbol{F}''$ sia quasi ovunque nullo, è che per un qualsiasi vettore $\boldsymbol{F}$ di norma integrale finita valga l'eguaglianza di Parseval: $\Sigma d^2_r = (\boldsymbol{F}, \boldsymbol{F})$.

Osserviamo anche che, se le componenti $\omega_{r,h}$ di $\boldsymbol{\omega}_r$ sono limitate, nel sistema (7) si possono supporre le incognite f_h sommabili nelle rispettive varietà I_h. Vale anche in tal caso la (10), dove $\boldsymbol{F}'$ è una particolare soluzione, di componenti sommabili, del sistema (7) e $\boldsymbol{F}''$ il più generale vettore, di componenti sommabili, ortogonale alla successione $\{\boldsymbol{\omega}_r\}$. La possibilità di ottenere il vettore $\boldsymbol{F}'$ come somma della serie $\Sigma d_r \boldsymbol{z}_r$, calcolata con un conveniente metodo di sommazione, richiede però un particolare studio, analogamente a quanto avviene per le serie di Fourier.

Ciò premesso, considerando sempre il problema di Dirichlet, possiamo enunciare la seguente proposizione.

c) *Sia* $\{w_z\}$ *una successione di funzioni di classe* 2 *in* τ *tale che la successione di vettori* $\{\boldsymbol{\omega}_r\}$, *di componenti* $-E^*(w_r)$ *in* τ, w_r *su* σ, *risulti chiusa. Detto* $\boldsymbol{G}$ *il vettore di componenti* u *in* τ, $\partial u/\partial \nu$ *su* σ, *si ricava dalla* (6) *il sistema di Fischer-Riesz*:

$$(\boldsymbol{G}, \boldsymbol{\omega}_r) = \int_\sigma u((\partial w_r/\partial \nu) - L w_r)\, d\sigma - \int_\tau f w_r\, d\tau. \tag{11}$$

Sono perciò noti i coefficienti di Fourier dell'incognito vettore $\mathbf{G}$ *rispetto alla successione* $\{\omega_r\}$ *è quindi tale vettore risulta individuato.*

Negli enunciati a), c) si suppone manifestamente che la soluzione esista e sia unica. Osserviamo infatti che, se manca il teorema di unicità non esistono successioni $\{v_r\}$, $\{w_r\}$ soddisfacenti alle condizioni di chiusura poste in a), c).

Indicheremo perciò le condizioni cui debbono soddisfare le successioni $\{v_r\}$, $\{w_r\}$ per poter trattare, attraverso i sistemi (5), (11), anche questi casi.

d) *Le successioni* $\{v_r\}$, $\{w_r\}$ *debbono esser tali che, se la soluzione del problema considerato non esiste, i sistemi* (5), (11) *non ammettano soluzioni. Se manca il teorema di unicità, le derivate conormali delle autosoluzioni debbono esser caratterizzate dall'essere ortogonali, su* σ, *alla successione* $\{v_r\}$, *le autosoluzioni dall'essere i corrispondenti vettori* $\mathbf{G}$ *ortogonali alla successione* $\{\omega_r\}$.

Proposizioni analoghe alle a), b), c), d), si possono enunciare per gli altri problemi al contorno relativi alla (1).

Si presenta ora la questione di costruire effettivamente le successioni $\{v_r\}$, $\{w_r\}$ *dotate delle proprietà richieste in* a), b), c), d), *di risolvere cioè i problemi porti dal Picone.*

A tale proposito, il Picone,[6] considerando le sole proposizioni a), b) ha dimostrato, come conseguenza della nota chiusura in τ della successione di funzioni $x_1^{\alpha_1} \cdots x_m^{\alpha_m}$, con $\alpha_i = 0, 1, \cdots$, rispetto alla totalità delle funzioni sommabili in τ, che, se la (1) è a coefficienti costanti e se $(p'_1, \cdots, p'_m)$ non è soluzione dell'equazione

$$\Sigma a_{ik} p_i p_k - \Sigma b_i p_i + c = 0, \tag{12}$$

posto $w = e^{\Sigma p_i x_i}$ e inoltre

$$w_r = \left(\frac{\partial^{\alpha_1 + \ldots + \alpha_m}}{\partial p_1^{\alpha_1} \cdots \partial p_m^{\alpha_m}} e^{\Sigma p_i x_i} \right)_{(p_i = p'_i)}, \qquad (\alpha_i = 0, 1, \cdots)$$

la successione $\{E^*(w_r)\}$ è chiusa rispetto alla totalità delle funzioni sommabili in τ.

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[6] Il Prof. Picone, dal quale gli eventi della guerra mi separarono, impedendomi per più di un anno di comunicare con Lui, mi partecipa ora di aver già Lui stesso assunto, per i problemi d'integrazione delle equazioni lineari a derivate parziali, di qualsivoglia ordine e tipo, il punto di vista al quale appartiene la proposizione c). Ciò formò oggetto di una sua conferenza—tenuta nel 1943 ma non pubblicata e della quale non ero al corrente durante la redazione del presente lavoro—avente per titolo: Questioni di topologia nella teoria dell'integrazione delle equazioni lineari a derivate parziali.

Quanto alla successione $\{v_r\}$, sempre supponendo la (1) a coefficienti costanti, dopo aver osservato che dalla funzione $v = e^{\sum p_i x_i}$, nell'ipotesi che i parametri p_i soddisfino alla (12), si può dedurre, in moda analogo a quanto si è fatto per la w (mediante le derivazioni di tutti gli ordini rispetto ai parametri p_i, effettuate tenendo conto della (12)) una successione $\{v_r\}$ di integrali della (4), ha proposto di studiare in general tale successione, rilevando che essa è sicuramente chiusa se la (1) è l'equazione di Laplace in 2 variabili e se τ è semplicemente connesso, perchè in tal caso si può prendere $v_r = (x_1 \pm x_2)^r$.

In un recente lavoro [7] ho risolto, nel caso generale, la questione posta dal Picone, dimostrando che la successione di funzioni $\{v_r\}$ da Lui indicata permette di risolvere i problemi al contorno per la (1), ma soltanto nell' ipotesi che il dominio τ abbia connessione ipersuperficiale semplice.

Nello stesso lavoro ho indicato un'ultra successione $\{v_r\}$ che consente la risoluzione dei problemi considerati, qualunque sia la connessione di τ. Il procedimento di integrazione da me seguito si basa sulla conoscenza della soluzione fondamentale della (4), ben nota trattandosi di un'equazione a coefficienti costanti, e si può estendre a numerosi altri problemi al contorno relativi a equazioni lineari; inoltre in esso non si richiede la conoscenza della successione $\{w_r\}$ considerata nella proposizione b).

Nella prima parte di questo lavoro è appunto indicato come tale procedimenta si applichi all'integrazione della (1), supponendola a coefficienti variabili.

Per questo cominciamo con l'ammettere che le funzioni a_{ik}, b_i, c si possano prolungare in un dominio τ' dotato di contorno σ' di classe 2, contenente τ nel suo interno, supponendo inoltre che in τ' le a_{ik}, b_i risultino di classe $2H$, $1H$ rispettivamente, la c di classe H.

Esiste allora [8] se R è un qualsiasi punto prefissato internamente a τ' e N un punto variabile in τ', la soluzione fondamentale $F(N,R)$ dell'equazione $E^*(u) = 0$ (che possiamo scegliere in modo da soddisfare alla $E(u) = 0$ come funzione di R) e valgono, per ogni integrale della (1) di classe 2 in τ, le formule fondamentali di Green:

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[7] M. Picone, "Nuovi metodi risolutivi per i problemi d'integrazione delle equazioni lineari a derivate parziali e nuova applicazione della trasformata multipla di Laplace nel caso delle equazioni a coefficienti costanti." *Rendiconti Accademia delle Scienze di Torino*, 1940, pp. 413-26.

[8] L. Amerio, "Sull'integrazione dell'equazione $\Delta_2 u - \lambda^2 u = f$ in un dominio di connessione qualsiasi," *Rendiconti Istituto Lombardo di Scienze e Lettere*, vol. 78 (1944-45).

$$(13)\qquad k_m u(P) = \int_\sigma \{u(M)((\partial F(M,P)/\partial \nu) - L(M)F(M,P)) - (\partial u(M)/\partial \nu)F(M,P)\}d\sigma - \int_\tau f(M)F(M,P)d\tau$$

$$(14)\qquad 0 \equiv \int_\sigma \{u(M)((\partial F(M,Q)/\partial \nu) - L(M)F(M,Q)) - (\partial u(M)/\partial \nu)F(M,Q)\}d\sigma - \int_\tau f(M)F(M,Q)d\tau$$

dove k_m è una costante non nulla dipendente solo da m, P e Q indicano rispettivamente un punto interno e un punto esterno a τ, M è il punto, di τ o di σ, rispetto alle cui coordinate si effettuano la integrazioni.

Noi considereremo sempre i problemi al contorno per la (1) *nella totalità degli integrali per cui valgano le* (13), (14) *intendendo, nel caso più generale, le funzioni* $u(M)$, $\partial u(M)/\partial \nu$ *definite quasi ovunque, su* σ, *come i limiti*

$$\lim_{P\to M} u(P), \qquad \lim_{P\to M} (\partial u(P)/\partial \nu),$$

dove $P \to M$ *lungo la conormale* ν *in* M, *e sommabili su* σ.

Se poniamo, su σ, $u(M) = A(M)$, $\partial u(M)/\partial \nu = B(M)$, dalla (14) segue che A, B soddisfano all'equazione

$$(15)\qquad 0 \equiv \int_\sigma \{A(M)((\partial F(M,Q)/\partial \nu) - L(M)F(M,Q)) - B(M)F(M,Q)\}d\sigma - \int_\tau f(M)F(M,Q)d\tau$$

la quale appare perciò come *necessaria* per le coppie A, B di funzioni che coincidono rispettivamente con i valori assunti su σ da un integrale u della (1) e della sua derivata conormale $\partial u/\partial \nu$. Ma tale equazione è anche *sufficiente*: ho dimostrato infatti, facendo sulla natura di σ delle ipotesi largamente verificate nei casi pratici, che *se* A, B *è una coppia di funzioni soddisfacenti alla* (15), *la funzione* $u(P)$ *definita internamente a* τ *dall'eguaglianza*

$$(16)\qquad k_m u(P) = \int_\sigma \{A(M)((\partial F(M,P)/\partial \nu) - L(M)F(M,P)) - B(M)F(M,P)\}d\sigma - \int_\tau f(M)F(M,P)d\tau$$

è un integrale della (1) *soddisfacente su* σ *alle condizioni*

$$u(M) = A(M), \qquad \partial u(M)/\partial \nu = B(M).$$

Si può perciò dire che la (15) rappresenta *l'equazione di compatibilità* tra i valori assunti su σ da un integrale della (1) e dalla sua derivata conormale.

La (15) *equivale* poi a un sistema di Fischer-Riesz. Sia infatti $\{g_r(Q)\}$ una successione di funzioni continue, chiusa nel dominio $\bar{\tau} = \tau' - \tau + \sigma$ rispetto alla totalità delle funzioni sommabili in $\bar{\tau}$ e continue nell'interno di $\bar{\tau}$, totalità alla quale appartengono gli integrali a secondo membro della (15) (si può, ad esempio, prendere $g_r(Q) = x_1^{\alpha_1} \cdots x_m^{\alpha_m}$, con $\alpha_i = 0, 1, \cdots$). La (15) equivale allora al sistema di infinite equazioni

$$\int_{\bar{\tau}} g_r(Q)\Big[\int_\sigma \{A(M)((\partial F(M;Q)/\partial \nu) - L(M)F(M,Q)) - B(M)F(M,Q)\}d\sigma - \int_\tau f(M)F(M,Q)d\tau\Big]d\bar{\tau} = 0$$

cioè anche, posto

$$v_r(M) = \int_{\bar{\tau}} g_r(Q)F(M,Q)d\bar{\tau}, \tag{17}$$

al sistema

$$\int_\sigma \{A(M)((\partial v_r(M)/\partial \nu) - L(M)v_r(M)) - B(M)v_r(M)\}d\sigma - \int_\tau f(M)v_r(M)d\tau = 0. \tag{18}$$

Se si considera, ad esempio, il problema di Dirichlet e se vale, in tale problema, per la (1), il teorema di unicità si ha perciò che la successione $\{v_r\}$ data dalla (17) è chiusa su σ e soddisfa, essendo $v_r(M)$ un integrale della (4), alle condizioni richieste nella proposizione a). Se non vale il teorema di unicità, le autosoluzioni sono *caratterizzate* dall'essere le corrispondenti funzioni $B(M)$ ortogonali su σ alla successione $\{v_r\}$. Infine, se il problema non ammette soluzione, nemmeno il sistema (18) ammette soluzione. La successione $\{v_r\}$ soddisfa perciò alle condizioni poste in d).

La (16) dà poi l'integrale della (1) corrispondente a una determinata coppia di funzioni A, B.

Il procedimenta di integrazione indicato non richiede l'uso della successione $\{w_r\}$ considerata nella proposizione b); in esso però si suppone esplicitamente nota la soluzione fondamentale $F(N,R)$ la cui effettiva costruzione è, in generale, assai ardua. Per ovviare a tale inconveniente ho indicato, nella seconda parte di questo lavoro un altro procedimento, in virtù del quale l'uso di tale funzione è completamente eliminato, essendo sufficiente di conoscerne l'esistenza, ciò che avviene se i coefficienti a_{ik}, b_i, c soddisfano alle condizioni dianzi ricordate.

Per questo, poniamo nella (3) in luogo di w la funzione

$$w_r = x_1^{\alpha_1} \cdots x_m^{\alpha_m} \qquad (\alpha_i = 0, 1, \cdots)$$

e, su σ, $u = A$, $\partial u/\partial \nu = B$. Otteniamo in tal modo il sistema di infinite equazioni di Fischer-Riesz:

$$(19) \qquad -\int_\tau u E^*(w_r)\, d\tau = \int_\sigma \{A(\partial w_r/\partial \nu - L w_r) - B w_r\} d\sigma - \int_\tau f w_r d\tau.$$

Anche tale sistema *caratterizza* gli integrali della (1). Ho dimostrato infatti che *se u, A, B sono tre funzioni per cui valgono le* (15), (16), *con A, B funzioni sommabili su σ, le funzioni u, A, B soddisfano al sistema* (19). *Viceversa, se u, A, B sono tre funzioni sommabili nei rispettivi insiemi di definizione e soddisfacenti al sistema* (19), *allora A, B soddisfano all'equazione* (15) *e u, ove se ne alteri al più il valore in un insieme di misura nulla, è data dalla* (16).

Per risolvere il problema di Dirichlet si procederà allora nel modo seguente. Si considerino le funzioni incognite u in τ, B in σ come componenti di un vettore $\boldsymbol{G}$, le funzioni $-E^*(w_r)$ in τ, w_r in σ come componenti di un vettore $\boldsymbol{\omega}_r$.

Si ottiene perciò che l'incognito vettore $\boldsymbol{G}$ soddisfa al sistema (11) in cui si ponga A in luogo di u. Se vale poi il teorema di unicità la successione $\{\boldsymbol{\omega}_r\}$ così ottenuta risulta chiusa e soddisfa alle condizioni poste in c). Se tale teorema non vale, le autosoluzioni sono *caratterizzate* dall'essere i corrispondenti vettori $\boldsymbol{G}$ ortogonali alla successione $\{\boldsymbol{\omega}_r\}$. Infine se manca la soluzione del problema di Dirichlet, anche il sistema (11) non ammette soluzione. La successione $\{\boldsymbol{\omega}_r\}$ soddisfa perciò alla condizioni poste in d).

Ragionamenti del tutto analoghi si fanno per gli altri problemi al contorno.

Osserviamo infine che, in luogo della successione $x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ si può assumere, sotto condizioni assai larghe, come successione $\{w_r\}$ una qualsiasi successione chiusa in τ' rispetto alla totalità delle funzioni ivi sommabili. In ogni caso la corrispondente successione $\{E^*(w_r)\}$ risulta chiusa in τ rispetto alla totalità delle funzioni sommabili in τ e soddisfa perciò alla condizioni posta in b).

I. Primo procedimento di integrazione.

1. Ricordiamo che la parte principale $V(M,P)$ della soluzione fondamentale $F(M,P)$ è data da [9]

$$V(M,P) = \{Z(M,P)\}^{-(m-2/2)}, \qquad \text{per } m > 2,$$

(20)

$$V(M,P) = -\tfrac{1}{2}\log Z(M,P), \qquad \text{per } m = 2,$$

con

(21) $$Z(M,P) = \Sigma A_{ik}(P)\,(x_i - \xi_i)\,(x_k - \xi_k)$$

dove $A_{ik}(P)$ è l'aggiunto di $a_{ik}(P)$, $(x_1, \cdots x_m)$ e $(\xi_1, \cdots, \xi_m)$ sono le coordinate di M e P rispettivamente.

Ciò premesso, dimostriamo un lemma relativo al comportamento di alcune derivate della funzione $F(M,P)$.

Per questo, portata l'origine Ω delle coordinate di S_m in un punto M_0 di σ, cominciamo col supporre che, in M_0, σ risulti di classe 2. Fatta coincidere la normale a σ in M_0 con l'asse x_m, in un conveniente intorno completo σ_0 di Ω. σ avrà perciò equazione $x_m = \phi(x_1, \cdots, x_{m-1})$, con ϕ funzione di classe 2. Sia λ la conormale in Ω, ν la conormale in un generico punto M di σ_0. Posto $a = \{\Sigma\nu^2_i(M_0)\}^{\frac{1}{2}}$, sia δ la distanza da Ω di un punto $P(\xi_1, \cdots, \xi_m)$ interno a τ e situato sulla conormale in Ω. Siccome la conormale in Ω pentra in τ, il punto Q simmetrico di P rispetto a Ω sarà esterno a τ, se la distanza δ è sufficientemente piccola. Poniamo poi $\lambda = \delta/a$.

Dette allora $\delta' = \lambda' a$, $\delta'' = \lambda'' a$ le ascisse di P e Q sulla conormale in Ω, sarà $\lambda' = \lambda$, $\lambda'' = -\lambda$ e quindi risulterà, in particolare,

$$\frac{\partial F(M,P)}{\partial \lambda'} = \frac{\partial F(M,P)}{\partial \lambda}, \quad \frac{\partial F(M,Q)}{\partial \lambda''} = -\frac{\partial F(M,Q)}{\partial \lambda},$$

$$\frac{\partial^2 F(M,Q)}{\partial \nu \partial \lambda'} - \frac{\partial^2 F(M,Q)}{\partial \nu \partial \lambda''} = \frac{\partial^2 F(M,P)}{\partial \nu \partial \lambda} + \frac{\partial^2 F(M,Q)}{\partial \nu \partial \lambda}.$$

LEMMA. *Posto* $s = \{\sum_1^{m-1} x^2_i\}^{\frac{1}{2}}$, *valgono per le derivate della soluzione fondamentale le limitazioni*

$$\frac{\partial F(M,P)}{\partial \nu} = O\left(\frac{\lambda}{(s^2+\lambda^2)^{m/2}} + \frac{1}{s^{m-2}}\right),$$

$$\frac{\partial F(M,P)}{\partial \lambda} = O\left(\frac{\lambda}{(s^2+\lambda^2)^{m/2}} + \frac{1}{s^{m-2}}\right),$$

$$\frac{\partial^2 F(M,P)}{\partial \nu \partial \lambda} + \frac{\partial^2 F(M,Q)}{\partial \nu \partial \lambda} = O\left(\frac{\lambda}{(s^2+\lambda^2)^{m/2}} + \frac{1}{s^{m-2}}\right)$$

[9] Vedasi, ad esempio, G. Ascoli, *loc. cit.*[2], pp. 70-76.

nelle quali, per $m = 2$, in luogo di $s^{-(m-2)}$, dovrà porsi $|\log s|$.

Dimostrazione. Supporremo $m > 2$, la dimostrazione per $m = 2$ essendo del tutto analoga.

a) Dimostriamo dapprima che per la funzione $V(M, P)$ data dalle (20), (21) valgono le limitazioni

$$(22)\qquad \begin{aligned} &\frac{\partial V(M,P)}{\partial \nu} = O\left(\frac{\lambda}{(s^2+\lambda^2)^{m/2}} + \frac{1}{s^{m-2}}\right), \\ &\frac{\partial V(M,P)}{\partial \lambda} = O\left(\frac{\lambda}{(s^2+\lambda^2)^{m/2}} + \frac{1}{s^{m-2}}\right), \\ &\frac{\partial^2 V(M,P)}{\partial \nu \partial \lambda} + \frac{\partial^2 V(M,Q)}{\partial \nu \partial \lambda} = O\left(\frac{\lambda}{(s^2+\lambda^2)^{m/2}} + \frac{1}{s^{m-2}}\right). \end{aligned}$$

Per questo, cominciamo col ricordare che risulta, per il parametro direttore ν_i della conormale ν,

$$(23)\qquad \nu_i = \Sigma a_{ik}(M) \cos(n x_k),$$

n essendo la normale in M, orientata verso l'interno di τ.

Si ha allora, per le (20), (21), ricordando che il determinante $\| a_{ik} \|$ è unitario,

$$(24)\qquad \begin{aligned} \frac{\partial V(M,P)}{\partial \nu} &= -(m-2) Z(M,P)^{-m/2} \Sigma A_{ik}(P)(x_k - \xi_k)\nu_i \\ &= -(m-2) Z(M,P)^{-m/2} \Sigma A_{ik}(P)(x_k - \xi_k) a_{il}(M) \cos(n x_l) \\ &= -(m-2) Z(M,P)^{-m/2} \{ \Sigma (x_k - \xi_k) \cos(n x_k) \\ &\quad + \Sigma (a_{il}(M) - a_{il}(P)) A_{ik}(P)(x_k - \xi_k) \cos(n x_l) \} \end{aligned}$$

e, posto $\rho = \{ \sum_1^m x^2_i \}^{\frac{1}{2}}$, $\cos\theta_i = x_i/\rho$, risulta

$$(25)\qquad \begin{aligned} Z(M,P) &= \Sigma A_{ik}(P)(x_i - \xi_i)(x_k - \xi_k) \\ &= \Sigma A_{ik}(P) x_i x_k + \Sigma A_{ik}(P)\xi_i \xi_k - 2\Sigma A_{ik}(P)\xi_i x_k \\ &= \rho^2 \Sigma A_{ik}(P) \cos\theta_i \cos\theta_k + \lambda^2 \Sigma A_{ik}(P) a_{im}(\Omega) a_{km}(\Omega) \\ &\quad - 2\lambda\rho \Sigma A_{ik}(P) a_{im}(\Omega) \cos\theta_k, \end{aligned}$$

essendo, per la (23),

$$(26)\qquad \xi_i = \lambda a_{im}(\Omega).$$

Si ha poi

(27) $\Sigma A_{ik}(P)a_{im}(\Omega)a_{km}(\Omega) = a_{mm}(\Omega) + \Sigma(A_{ik}(P) - A_{ik}(\Omega))a_{im}(\Omega)a_{km}(\Omega)$

ed è $a_{mm}(\Omega) \neq 0$ perchè la forma quadratica $\Sigma a_{ik}\xi_i\xi_k$ è definita positiva.

E' infine

(28) $\Sigma A_{ik}(P)a_{im}(\Omega)\cos\theta_k = \cos\theta_m + \Sigma(A_{ik}(P) - A_{ik}(\Omega))a_{im}(\Omega)\cos\theta_k$

e siccome risulta, indicando con α una costante positiva,

(29) $$|x_m| = |\phi(x_1, \cdots, x_{m-1})| \leq \alpha s^2,$$

(30) $$\lim \rho/s = 1,$$

si ricava

(31) $$|\cos\theta_m| \leq \beta s$$

con β costante positiva.

Dalle (25), (27), (28), (29), (30), (31), essendo la forma $\Sigma A_{ik}\xi_i\xi_k$ definita positiva, segue perciò che è possibile determinare $\delta > 0$ in modo che, per $|x_i| \leq \delta$, $0 < \lambda \leq \delta$, risulti

(32) $$p(s^2 + \lambda^2) \leq Z(M, P) \leq q(s^2 + \lambda^2)$$

con $0 < p < q$.

Siccome poi si ha

(33) $$|\cos(nx_k)| \leq \gamma s, \qquad (k = 1, \cdots, m-1)$$

con γ costante positiva, si ricava, per le (26), (29),

$$|\Sigma(x_k - \xi_k)\cos(nx_k)| \leq \eta(s^2 + \lambda),$$

dove η è una costante positiva, e quindi dalle (24), (32) segue la prima delle (22).

Consideriamo ora la derivata $\partial V(M, P)/\partial\lambda$. Si ha, per la (26), posto $A_{ikh}(P) = \partial A_{ik}(P)/\partial\xi_h$,

(34) $$\frac{\partial V(M,P)}{\partial\lambda} = -\frac{m-2}{2} Z(M,P)^{-m/2}\{-2\Sigma a_{hm}(\Omega)A_{hk}(P)(x_k - \xi_k) + \Sigma a_{hm}(\Omega)A_{ikh}(P)(x_k - \xi_k)(x_i - \xi_i)\}$$

$$= (m-2)Z(M,P)^{-m/2}(x_m - \xi_m) - \frac{m-2}{2} Z(M,P)^{-m/2}\{-2\Sigma a_{hm}(\Omega)(A_{hk}(P) - A_{hk}(\Omega))(x_k - \xi_k) + \Sigma a_{hm}(\Omega)A_{ikh}(P)(x_k - \xi_k)(x_i - \xi_i)\}$$

e quindi dalle (26), (29), (32) segue la seconda delle (22).

Studiamo infine il comportamento della derivata $\partial^2 V(M,P)/\partial\nu\partial\lambda$.
Si ha, per la (26),

$$(35)\quad \frac{\partial^2 V(M,P)}{\partial\nu\partial\lambda} = \frac{m(m-2)}{2} Z(M,P)^{-(m+2/2)}\{-2\Sigma a_{hm}(\Omega)A_{hk}(P)(x_k-\xi_k)$$
$$+\Sigma a_{hm}(\Omega)A_{ikh}(P)(x_k-\xi_k)(x_i-\xi_i)\}\{\Sigma a_{il}(M)A_{ik}(P)(x_k-\xi_k)\cos(nx_l)\}$$
$$-(m-2)Z(M,P)^{-m/2}\{-\Sigma a_{hm}(\Omega)a_{il}(M)A_{ih}(P)\cos(nx_l)$$
$$+\Sigma a_{hm}(\Omega)a_{il}(M)A_{ikh}(P)(x_k-\xi_k)\cos(nx_l)\}$$

da cui, per le (24), (34),

$$(36)\quad \frac{\partial^2 V(M,P)}{\partial\nu\partial\lambda} = \frac{m(m-2)}{2} Z(M,P)^{-(m+2/2)}\{-2(x_m-\xi_m)$$
$$-2\Sigma a_{hm}(\Omega)(A_{hk}(P)-A_{hk}(\Omega))(x_k-\xi_k)$$
$$+\Sigma a_{hm}(\Omega)A_{ikh}(P)(x_k-\xi_k)(x_i-\xi_i)\}\{\Sigma(x_k-\xi_k)\cos(nx_k)$$
$$+\Sigma a_{il}(M)(A_{ik}(P)-A_{ik}(M))(x_k-\xi_k)\cos(nx_l)\}$$
$$-(m-2)Z(M,P)^{-m/2}\{-\Sigma a_{im}(M)\cos(nx_l)$$
$$+\Sigma a_{hm}(\Omega)a_{il}(M)A_{ikh}(\Omega)(x_k-\xi_k)\cos(nx_l)$$
$$-\Sigma a_{hm}(\Omega)a_{il}(M)(A_{ih}(P)-A_{ih}(\Omega))\cos(nx_l)$$
$$+\Sigma a_{hm}(\Omega)a_{il}(M)(A_{ikh}(P)-A_{ikh}(\Omega))(x_k-\xi_k)\cos(nx_l)\}$$
$$=-m(m-2)Z(M,P)^{-(m+2/2)}\{(x_m-\xi_m)(\Sigma(x_k-\xi_k)\cos(nx_k))+\eta\}$$
$$-(m-2)Z(M,P)^{-m/2}\{-\Sigma a_{im}(M)\cos(nx_l)$$
$$+\Sigma a_{hm}(\Omega)a_{il}(M)A_{ikh}(\Omega)(x_k-\xi_k)\cos(nx_l)\}$$

e risulta, per le (29), (32), (33),

$$(37)\quad \eta Z(M,P)^{-(m+2/2)} = O((s^2+\lambda^2)^{-(m+2/2)}[(s^2+\lambda)(s+\lambda)^2$$
$$+(\lambda(s+\lambda)+(s+\lambda)^2)(s^2+\lambda+(s+\lambda)^2)])$$
$$= O\left(\frac{\lambda}{(s^2+\lambda^2)^{m/2}}+\frac{1}{s^{m-2}}\right),$$

$$(38)\quad \xi Z(M,P)^{-m/2} = O((s^2+\lambda^2)^{-m/2}[\lambda+\lambda(s+\lambda)])$$
$$= O\left(\frac{\lambda}{(s^2+\lambda^2)^{m/2}}+\frac{1}{s^{m-2}}\right).$$

Consideriamo ora la derivata $\partial^2 V(M,Q)/\partial\nu\partial\lambda$. Siccome le coordinate del punto Q sono $\eta_i = -\xi_i = -\lambda a_{im}(\Omega)$, risulta

$$\frac{\partial^2 V(M,Q)}{\partial\nu\partial\lambda} = -\Sigma a_{im}(\Omega)\frac{\partial^2 V(M,Q)}{\partial\nu\partial\eta_i}$$

e quindi si ricava dalla (36)

$$(39)\quad \frac{\partial^2 V(M,Q)}{\partial\nu\partial\lambda} = m(m-2)Z(M,Q)^{-(m+2/2)}(x_m+\xi_m)\Sigma(x_k+\xi_k)\cos(nx_i)$$
$$+ (m-2)Z(M,Q)^{-m/2}\{-\Sigma a_{lm}(M)\cos(nx_l)$$
$$+ \Sigma a_{hm}(\Omega)a_{il}(M)A_{ikh}(\Omega)(x_k+\xi_k)\cos(nx_l)\} + \mu$$

dove μ soddisfa, per le (37), (38), alla limitazione

$$(40)\qquad \mu = O\left(\frac{\lambda}{(s^2+\lambda^2)^{m/2}} + \frac{1}{s^{m-2}}\right).$$

Si ha poi, per le (36), (39),

$$(41)\quad \frac{\partial^2 V(M,P)}{\partial\nu\partial\lambda} + \frac{\partial^2 V(M,Q)}{\partial\nu\partial\lambda}$$
$$= -m(m-2)\{Z(M,P)^{-(m+2/2)} - Z(M,Q)^{-(m+2/2)}\}\xi_m\Sigma\xi_k\cos(nx_k)$$
$$+ (m-2)\{Z(M,P)^{-m2} - Z(M,Q)^{-m/2}\}\{\Sigma a_{lm}(M)\cos(nx_l)$$
$$- \Sigma a_{hm}(\Omega)a_{il}(M)A_{ikh}(\Omega)x_k\cos(nx_l)\} + \psi$$

e risulta, per le (29), (33), (36), (37), (38), (39), (40),

$$(42)\qquad \psi = O\left\{\frac{\lambda}{(s^2+\lambda^2)^{m/2}} + \frac{1}{s^{m-2}}\right\}.$$

Si ha inoltre

$$Z(M,P) - Z(M,Q) = \Sigma a_{ik}(P)(x_i-\xi_i)(x_k-\xi_k) - \Sigma A_{ik}(Q)(x_i+\xi_i)(x_k+$$
$$= \Sigma(A_{ik}(P) - A_{ik}(Q))(x_ix_k+\xi_i\xi_k) - 2\Sigma(A_{ik}(P) + A_{ik}(Q))x_i\xi_k$$

e quindi, osservando che risulta, per la (26),

$$\Sigma(A_{ik}(P) + A_{ik}(Q))x_i\xi_k = 2\lambda x_m + \lambda\Sigma(A_{ik}(P) + A_{ik}(Q) - 2A_{ik}(\Omega))x_i a_{km}($$

si ricava

$$Z(M,P) - Z(M,Q) = O(\lambda(s^2+\lambda^2)).$$

Ne segue, per h intero positivo e par la (32),

$$Z(M,P)^h - Z(M,Q)^h = (Z(M,P) - Z(M,Q))\sum_0^{h-1} Z(M,P)^i Z(M,Q)^{h-1-i}$$
$$= O(\lambda(s^2+\lambda^2)^h).$$

Si ha perciò

$$(43)\quad Z(M,P)^{-h/2} - Z(M,Q)^{-h/2} = \frac{Z(M,Q)^h - Z(M,P)^h}{(Z(M,P)^{h/2} + Z(M,Q)^{h/2})Z(M,P)^{h/2}Z(M,Q)^{h/2}}$$
$$= O\left(\frac{\lambda}{(s^2+\lambda^2)^{h/2}}\right).$$

e quindi dalle (41), (42), (43) segue

$$\frac{\partial^2 V(M,P)}{\partial\nu\partial\lambda} + \frac{\partial^2 V(M,Q)}{\partial\nu\partial\lambda} = O\left(\frac{\lambda^3}{(s^2+\lambda^2)^{m+2/2}} + \frac{\lambda}{(s^2+\lambda^2)^{m/2}} + \frac{1}{s^{m-2}}\right)$$
$$= O\left(\frac{\lambda}{(s^2+\lambda^2)^{m/2}} + \frac{1}{s^{m-2}}\right).$$

Perciò anche la terza della (22) è dimostrata.

b) Ricordiamo che, per ipotesi, i coefficienti a_{ik}, b_i, c della (1) si possono prolungare in un dominio τ' contenente τ al suo interno (in modo da risultare in τ' di classi $2H$, $1H$, H rispettivamente) e consideriamo la funzione

$$(44)\qquad W(S,R) = \int_{\tau'} V(S,N)\,E^*_N(V(N,R))\,d\tau'$$

dove S, R sono due punti qualsiasi di τ' e si è indicato, per maggior chiarezza, con E^*_N l'operatore differenziale

$$\Sigma a_{ik}(N)\,(\partial^2/\partial y_i\partial y_k) + \Sigma b^*_i(N)\,(\partial/\partial y_i) + c^*(N),$$

$(y_1, \cdots, y_m)$ essendo le coordinate del punto N.

Se M, P, Q indicano ancora gli stessi punti considerati in a), dimostriamo che valgono le limitazioni

$$(45)\qquad \frac{\partial W(M,P)}{\partial\nu} = O\left(\frac{1}{s^{m-2}}\right),$$
$$\frac{\partial W(M,P)}{\partial\lambda} = O\left(\frac{1}{s^{m-2}}\right),$$
$$\frac{\partial^2 W(M,P)}{\partial\nu\partial\lambda} = O\left(\frac{\lambda}{(s^2+\lambda^2)^{m/2}} + \frac{1}{s^{m-2}}\right)$$

e le analoghe ottenute scambiando P con Q.

Cominciano col ricordare una nota proprietà della funzione $V(N,R)$. Si ha $V(N,R) = Z(N,R)^{-(m-2/2)}$ e quindi, indicate con $(\zeta_1, \cdots, \zeta_m)$ le coordinate di R,

$$(46)\quad \frac{\partial V(N,R)}{\partial y_i} = -(m-2)Z(N,R)^{-m/2}\Sigma A_{ir}(R)\,(y_r - \zeta_r),$$

$$(47)\quad \frac{\partial^2 V(N,R)}{\partial y_i\partial y_k} = m(m-2)Z(N,R)^{-(m+2/2)}\Sigma A_{ks}(R)A_{ir}(R)\,(y_s-\zeta_s)(y_r-\zeta_r)$$
$$-(m-2)Z(N,R)^{-m/2}A_{ik}(R),$$

da cui

$$(48)\quad \Sigma a_{ik}(N)\frac{\partial^2 V(N,R)}{\partial y_i \partial y_k} = m(m-2)Z(N,R)^{-(m+2/2)}\Sigma a_{ik}(N)A_{ks}(R)A_{ir}(R) \times (y_s-\zeta_s)(y_r-\zeta_r) - (m-2)Z(N,R)^{-m/2}\Sigma a_{ik}(N)A_{ik}(R) = m(m-2)Z(N,R)^{-(m+2/2)}\Sigma(a_{ik}(N)-a_{ik}(R)) \times A_{ks}(R)A_{ir}(R)(y_s-\zeta_s)(y_r-\zeta_r) - (m-2)Z(N,R)^{-m/2}\Sigma(a_{ik}(N)-a_{ik}(R))A_{ik}(R).$$

Ne seguono le note limitazioni

$$(49)\quad \Sigma a_{ik}(N)\frac{\partial^2 V(N,R)}{\partial y_i \partial y_k} = O(NR^{-(m-1)}),$$

$$(50)\quad E^*_N(V(N,R)) = O(NR^{-(m-1)}).$$

Siccome è anche

$$(51)\quad \frac{\partial V(M,N)}{\partial \nu} = O(MN^{-(m-1)}),$$

si deduce, per la (32),

$$\frac{\partial W(M,P)}{\partial \nu} = \frac{\partial V(M,N)}{\partial \nu} E^*_N(V(N,P))d\tau' = O(MP^{-(m-2)}) = O(Z(M,P)^{-(m-2/2)} = O(1/s^{m-2})$$

cioè la prima delle (45) è dimostrata.

Per studiare la derivata $\partial W(M,P)/\partial\lambda$ trasformiamo la (44) con la formula di Green. Per questo, si circondino i punti M e P con due ipersfere σ'_M, σ'_P aventi i centri rispettivamente in M, P e raggio ϵ; siano σ' il contorno di τ', τ'_M e τ'_P i domini ipersferici limitati da σ'_M, σ'_P.

Risulta, per la formula di Green, detta ν' la conormale in N, essendo E_N l'operatore differenziale aggiunto di E^*_N,

$$(52)\quad \int_{\tau'-\tau'_M-\tau'_P} V(M,N)E^*_N(V(N,P))d\tau' = \int_{\tau'-\tau'_M-\tau'_P} V(N,P)E_N(V(M,N))d\tau' - \int_{\sigma'}\{V(M,N)\frac{\partial V(N,P)}{\partial \nu'} - V(N,P)\frac{\partial V(M,N)}{\partial \nu'} - L(N)V(M,N)V(N,P)\}d\sigma' - \int_{\sigma'_M}\{V(M,N)\frac{\partial V(N,P)}{\partial \nu'} - V(N,P)\frac{\partial V(M,N)}{\partial \nu'} - L(N)V(M,N)V(N,P)\}d\sigma'_M - \int_{\sigma'_P}\{V(M,N)\frac{\partial V(N,P)}{\partial \nu'} - V(N,P)\frac{\partial V(M,N)}{\partial \nu'} - L(N)V(M,N)V(N,P)\}d\sigma'_P$$

Siccome, per $\epsilon \to 0$, i due ultimi integrali tendono rispettivamente a $\mp k_m V(M,P)$, segue dalla (52)

$$(53)\quad W(M,P) = \int_{\tau'} V(N,P) E_N(V(M,N))\, d\tau'$$
$$- \int_{\sigma'} \{V(M,N)\, \frac{\partial V(N,P)}{\partial \nu'} - V(N,P)\, \frac{\partial V(M,N)}{\partial \nu'} - L(N) V(M,N) V(N,P)\}\, d\sigma'$$

da cui si deduca la seconda delle (45).

Consideriamo infine la derivata $\partial^2 W(M,P)/\partial\nu\partial\lambda$. Per la (2), posto

$$(54)\quad \bar{E}^*_N = E^*_N - c^*,$$

si ricava

$$(55)\quad \frac{\partial^2 W(M,P)}{\partial\nu\partial\lambda} = \frac{\partial}{\partial\lambda} \int_{\tau'} \frac{\partial V(M,N)}{\partial\nu}\, \bar{E}^*_N(V(N,P))\, d\tau'$$
$$+ \int_{\tau'} \frac{\partial V(M,N)}{\partial\nu} c^*(N)\, \frac{\partial V(N,P)}{\partial\lambda}\, d\tau'$$

e risulta

$$(56)\quad \int_{\tau'} \frac{\partial V(M,N)}{\partial\nu}\, c^*(N)\, \frac{\partial V(N,P)}{\partial\lambda}\, d\tau' = O(MP^{-(m-2)}) = O(1/s^{m-2}).$$

Osserviamo ora che i punti M, P, Ω sono interni a τ'; possiamo perciò considerarli interni a un dominio ipersferico τ_0 interno a τ', con centro in Ω e raggio r, e siccome la funzione

$$(57)\quad U(M,P) = \int_{\tau'-\tau_0} V(M,N) \bar{E}^*_N(V(N,P)\, d\tau'$$

è continua in Ω insieme alle sue derivate $\partial u/\partial\nu$, $\partial u/\partial\lambda$, $\partial^2 U/\partial\nu\partial\lambda$, basterà dimostrare che risulta

$$(58)\quad \frac{\partial}{\partial\lambda} \int_{\tau_0} \frac{\partial V(M,N)}{\partial\nu}\, \bar{E}^*_N(V(N,P))\, d\tau_0 = O\left(\frac{\lambda}{(s^2+\lambda^2)^{m/2}} + \frac{1}{s^{m-2}}\right).$$

Poniamo ora $\delta = Z(M,P)^{\frac{1}{2}}$ e facciamo il cambiamento di variabili $x_i = \delta\bar{x}_i$, $y_i = \delta\bar{y}_i$, $\xi_i = \delta\bar{\xi}_i$. Indicato con $\bar{R} = R/\delta$ il punto che corrisponde, per la trasformazione effettuata, al punto R, si ha

$$(59)\quad V(M,N) = (1/\delta^{m-2}) \{\Sigma A_{ik}(\delta\bar{N})(\bar{x}_i - \bar{y}_i)(\bar{x}_k - \bar{y}_k)\}^{-(m-2/2)}$$
$$= \frac{\bar{V}(\bar{M},\bar{N},\delta)}{\delta^{m-2}} = \frac{\bar{Z}(\bar{M},\bar{N},\delta)^{-(m-2/2)}}{\delta^{m-2}}.$$

Ne segue, osservando che la normale $\bar{n}$ e la conormale $\bar{\nu}$ in $\bar{M}$ hanno gli stessi coseni direttori della normale n e della conormale ν in M,

$$(60)\qquad \frac{\partial V(M,N)}{\delta^{m-2}} = \frac{1}{\delta^{m-1}}\,\frac{\partial \bar V(\bar M,\bar N,\delta)}{\partial \bar\nu}$$

e inoltre, per le (46), (48) e (54),

$$(61)\qquad \bar E^*_N(V(N,P)) = \frac{1}{\delta^{m-1}}\{m(m-2)\bar Z(\bar N,\bar P,\delta)^{-(m+2/2)}\Sigma\frac{a_{ik}(\delta\bar N)-a_{ik}(\delta\bar P)}{\delta}$$
$$\times A_{ir}(\delta\bar P)(A_{ks}(\delta\bar P)(\bar y_r-\bar\xi_r)(\bar y_s-\bar\xi_s)$$
$$-(m-2)Z(\bar N,\bar P,\delta)^{-m/2}[\Sigma\frac{a_{ik}(\delta\bar N)-a_{ik}(\delta\bar P)}{\delta}$$
$$\times A_{ik}(\delta P)+\Sigma b^*_i(\delta\bar N)A_{ik}(\delta P)(\bar y_k-\bar\xi_k)]\}$$
$$=\frac{\psi(\bar N,\bar P,\delta)}{\delta^{m-1}}.$$

Ora, se $g(N)$ è una funzione di classe 2 per $\Omega N\leq r$, si ha

$$(62)\qquad \frac{\partial g(\delta\bar N)}{\partial\delta}=\Sigma\bar y_i\frac{\partial g(N)}{\partial y_i}=O(\bar\Omega\bar N),$$

$$(63)\qquad \frac{\partial}{\partial\delta}\,\frac{g(\delta\bar N)-g(\delta\bar P)}{\partial\delta}$$
$$=\frac{(\Sigma\delta\bar y_i(\partial g(N)/\partial y_i)-g(N))-(\Sigma\delta\bar\xi_i(\partial g(P)/\partial\xi_i)-g(P))}{\delta^2}$$
$$=\frac{(\Sigma y_i(\partial g(N)/\partial y_i)-g(N))-(\Sigma\xi_i(\partial g(P)/\partial\xi_i)-g(P))}{\delta^2}$$
$$=\Sigma(\bar y_h-\bar\xi_h)\bar l_i(\partial^2 g(T)/\partial t_i\partial t_k)=O((\bar\Omega\bar N+\Omega\bar P)\bar N\bar P).$$

essendo $T(t_1,\cdots,t_m)$ un punto del segmento NP.

Si ha poi, per le (59), (60), (61),

$$(64)\qquad \frac{\partial\bar V(\bar M,\bar N,\delta)}{\partial\bar\nu}=-(m-2)\bar Z(\bar M,\bar N,\delta)^{-m/2}\Sigma A_{ik}(\delta\bar N)(\bar x_k-\bar y_k)a_{il}$$
$$\times(\delta\bar M)\cos(\bar n\bar x_l)$$
$$=O(\bar M\bar N^{-(m-1)}),$$

$$(65)\qquad \psi(\bar N,\bar P,\delta)=O(\bar N\bar P^{-(m-1)}).$$

Inoltre, se h è un intero positivo, risulta, per le (62), (63), (64),

$$(66)\qquad \frac{\partial\bar Z(\bar N,\bar P,\delta)^{-h/2}}{\partial\delta}=-(h/2)\bar Z(\bar N,\bar P,\delta)^{-(h+2/2)}\Sigma(\partial A_{ik}(\delta\bar P)/\partial\delta)(\bar y_i-\bar\xi_i)$$
$$=O(\bar\Omega\bar P/\bar N\bar P^h)=O(\bar\Omega\bar P+\bar\Omega\bar N/\bar N\bar P^h),$$

$$(67)\quad \frac{\partial^2 \bar V(\bar M,\bar N,\delta)}{\partial\bar\nu\,\partial\delta} = \frac{m(m-2)}{2}\,\bar Z(\bar M,\bar N,\delta)^{-(m+2/2)}\sum\frac{\partial A_{ik}(\delta\bar N)}{\partial\delta}$$
$$\times(\bar x_i-\bar y_k)(\bar x_k-\bar y_k)A_{rs}(\delta\bar N)(\bar x_r-\bar y_r)a_{sl}(\delta\bar M)\cos(\bar n\bar x_l)$$
$$-(m-2)\bar Z(\bar M,\bar N,\delta)^{-m/2}\sum(\bar x_k-\bar y_k)\cos(\bar n\bar x_l)$$
$$\times\frac{\partial A_{ik}(\partial\bar N)}{\partial\delta}\,a_{il}(\delta\bar M)+A_{ik}(\delta\bar N)\frac{\partial a_{il}(\delta\bar M)}{\partial\delta}$$
$$= O\,\frac{\bar\Omega\bar N+\bar\Omega\bar M}{\bar M\bar N^{m-1}}$$

$$(68)\quad \frac{\partial\psi(\bar N,\bar P,\delta)}{\partial\delta} = O\left(\frac{\bar\Omega\bar N+\bar\Omega\bar P}{\bar N\bar P^{m-1}}\right).$$

Per la trasformazione effettuata, al dominio ipersferico τ_0, di raggio r e centro Ω, viene a corrispondere, in $\bar S_m$, un dominio ipersferico $\bar\tau_0$, di raggio $\bar r = r/\delta$ e centro $\bar\Omega$. Supponiamo inoltre che M e P abbiano da Ω distanza $\le r/2$; perciò sarà anche $\bar\Omega\bar M \le \bar r/2$, $\bar\Omega\bar P \le \bar r/2$.

Si ha poi, per le (60), (61),

$$\frac{\partial}{\partial\lambda}\int_{\tau_0}\frac{\partial V(M,N)}{\partial\nu}\,\bar E^*{}_N(V(N,P)\,d\tau_0 = \frac{\partial}{\partial\lambda}\left(\frac{1}{\partial^{m-2}}\int_{\bar\tau_0}\frac{\partial\bar V(\bar M,\bar N,\delta)}{\partial\bar\nu}\,\psi(\bar N,\bar P,\delta)\,d\bar\tau_0\right)$$
$$-\{-\frac{m-2}{\partial^{m-1}}\int_{\bar\tau_0}\frac{\partial\bar V(\bar M,\bar N,\delta)}{\partial\bar\nu}\,\psi(\bar N,\bar P,\delta)\,d\bar\tau_0$$
$$+\frac{1}{\partial^{m-2}}\frac{\partial}{\partial\delta}\int_{\bar\tau_0}\frac{\partial\bar V(\bar M,\bar N,\delta)}{\partial\bar\nu}\,\psi(\bar N,\bar P,\delta)\,d\bar\tau_0\}\,\frac{\partial\delta}{\partial\lambda}$$

da cui, osservando che, per il cambiamento di variabili effettuato, la distanza $\bar M\bar P$ soddisfa a una limitazione del tipo

$$0 < h \le \bar M\bar P \le k < \infty,$$

si ricava, per le (64), (65),

$$(70)\quad \int_{\bar\tau_0}\frac{\partial\bar V(\bar M,\bar N,\delta)}{\partial\bar\nu}\,\psi(\bar N,\bar P,\delta)\,d\bar\tau_0$$
$$= O\Big(\int_{-\infty}^{\infty}d\bar y_1\cdots\int_{-\infty}^{\infty}\bar M\bar N^{-(m-1)}\bar N\bar P^{-(m-1)}d\bar y_m\Big) = O(1).$$

Detta poi $\bar\sigma_0$ l'ipersfera che delimita $\bar\tau_0$, si ha

$$(71)\quad \frac{\partial}{\partial\delta}\int_{\bar\tau_0}\frac{\partial\bar V(\bar M,\bar N,\delta)}{\partial\bar\nu}\,\psi(\bar N,\bar P,\delta)\,d\bar\tau_0 = \int_{\bar\tau_0}\frac{\partial}{\partial\delta}\left(\frac{\partial\bar V(\bar M,\bar N,\delta)}{\partial\bar\nu}\right)\psi(\bar N,\bar P,\delta)\,d\bar\tau_0$$
$$-\frac{r}{\delta^2}\int_{\bar\sigma_0}\frac{\partial\bar V(\bar M,\bar N,\delta)}{\partial\bar\nu}\,\psi(\bar N,\bar P,\delta)\,d\bar\sigma_0$$

e, per le (64), (65), (67), (68), risulta, ricordando che su $\bar{\sigma}_0$ è $\bar{M}\bar{N} \geq \bar{r}/2$, $\bar{P}\bar{N} \geq \bar{r}/2$,

$$(72) \qquad \int_{\bar{\tau}_0} \frac{\partial}{\partial \delta}\left(\frac{\partial \bar{V}(\bar{M},\bar{N},\delta)}{\partial \bar{\nu}}\psi(\bar{N},\bar{P},\delta)\right)d\bar{\tau}_0$$

$$= O(1/\delta \int_{-\infty}^{\infty} d\bar{y}_1 \cdots \int_{-\infty}^{\sigma_0} \bar{M}\bar{N}^{-(m-1)}\bar{N}\bar{P}^{-(m-1)}d\bar{y}_m) = O(1/\delta),$$

$$(73) \qquad \frac{r}{\partial^2}\int_{\bar{\sigma}_0} \frac{\partial \bar{V}(\bar{M},\bar{N},\delta)}{\partial \bar{\nu}}\psi(\bar{N},\bar{P},\delta)d\bar{\sigma}_0 = O(1/\delta^2 \int_{\delta_0} \delta^{2(m-1)}d\bar{\sigma}_0) = O(\delta^{m-3}).$$

Dalle (69), (70), (71), (72), (73) si ricava perciò

$$(74) \qquad \frac{\partial}{\partial \lambda}\int_{\bar{\tau}_0} \frac{\partial V(M,N)}{\partial \nu}\bar{E}^*_N(V(N,P))d\bar{\tau}_0 = O\left(\frac{1}{\partial^{m-1}}\frac{\partial \delta}{\partial \lambda}\right).$$

Ora si ha

$$\delta^2 = \Sigma A_{ik}(P)(x_i - \xi_i)(x_k - \xi_k)$$

e quindi, per la (26),

$$(75) \quad 2\delta \frac{\partial \delta}{\partial \lambda} = \Sigma \frac{\partial A_{ik}(P)}{\partial \lambda}(x_i - \xi_i)(x_k - \xi_k) + 2\lambda \Sigma A_{ik}(P)a_{im}(\Omega)a_{km}(\Omega)$$

$$- 2\Sigma A_{ik}(P)x_i a_{km}(\Omega) = O(s^2 + \lambda)$$

come segue dalle (27), (28), (29), (31).

Dalle (55), (56), (57), (74), (75), segue allora la terza delle (45).

c) Dimostriamo ora, valendoci delle (22) e (45), le analoghe limitazioni per la soluzione fondamentale della (4).

Come è noto, E. E. Levi [10] ha posto tale funzione nella forma

$$(76) \qquad F(N,P) = V(N,P) + \int_{\tau'} V(N,T)\phi(T,P)d\tau' + \psi(N,P)$$

con ϕ, ψ funzioni opportune. Per determinarle si impone alla $F(N,P)$ di soddisfare, come funzione di N, all'equazione (4).

Detta σ_m la misura ipersuperficiale dell'ipersfera di raggio 1 in S_m, si ricava l'equazione integrale di Fredholm

$$(77) \quad \sigma_m \phi(N,P) = \int_{\tau'} E^*_N(V(N,T))\phi(T,P)d\tau'$$

$$+ E^*_N(V(N,P)) + E^*_N(\psi(N,P)).$$

[10] G. Ascoli, *loc. cit.*[2], p. 72.

Se il determinante della (77) è $\neq 0$, si prende $\psi(N,P) \equiv 0$; in caso contrario, per fare in modo che la (77) abbia soluzione, il Levi pone

$$\psi(N,P) = \Sigma u_i(N)\eta_i(P)$$

dove le $u_i(N)$ sono opportune funzioni di classe 2 in τ' e le $\eta_i(P)$ risultano combinazioni lineari delle autofunzioni $\psi_1(P), \cdots, \psi_k(P)$ del nucleo associato $(1/\sigma_m)E^*_T(V(T,N))$:

$$\psi_i(P) = 1/\sigma_m \int_{\tau'} E^*_T(V(T,P))\psi_i(T)\,d\tau'. \tag{78}$$

Ciò premesso, osserviamo che dalla (77), posto

$$K(N,T) = (1/\sigma_m)E^*_N(V(N,T)), \qquad \chi(N,P) = (1/\sigma_m)E^*_N(\psi(N,P)),$$

segue

$$\phi(N,P) - K(N,P) = \int_{\tau'} K(N,T)(\phi(T,P) - K(T,P))\,d\tau' + K_2(N,P) + \chi(N,P) \tag{79}$$

essendo $K_2(N,P)$ il primo nucleo iterato di $K(N,P)$.

Detto $H(N,T)$ il nucleo risolvente della (79), risulta allora

$$\phi(N,P) = K(N,P) + K_2(N,P) + \chi(N,P) + \int_{\tau'} H(N,T)(K_2(T,P) + \chi(T,P))\,d\tau'. \tag{80}$$

Ora, ragionando come in b), si ricava

$$K_2(T,P) = O(TP^{-(m-2)}),$$

$$\frac{\partial K_2(T,P)}{\partial \lambda} = O(TP^{-(m-1)}) \tag{81}$$

ed è inoltre

$$H(N,T) = O(NT^{-(m-1)}). \tag{82}$$

Siccome poi, per la (78), le funzioni $\psi_i(P)$ soddisfano anche all'equazione

$$\psi_i(P) = \int_{\tau'} K_{m+1}(T,P)\psi_i(T)\,d\tau'$$

e risultano perciò di classe 1 (essendo il nucleo iterato $K_{m+1}(T,P)$ dotato di derivate parziali rispetto alle coordinate di P continue per $P \neq T$, limitate in τ') anche la funzione $\chi(N,P)$ è di classe 1, come funzione di P.

La tesi [11] segue allora dalle (22), (45) e dalle (79), (80), (81), (82).

2. In quel che segue faremo le seguenti ipotesi, largamente verificate nei casi pratici, sulla natura del dominio limitato τ e delle funzioni A, B che compaiono nelle (15), (16).

Supporremo che il contorno σ abbia misura ipersuperficiale finita e risulta di classe 2 in ogni suo punto, esclusi al più i punti di un insieme chiuso χ di misura ipersuperficiale nulla. Ammetteremo inoltre che valga in τ la formula di Green per ogni coppia u, w di funzioni di classe 2. Si noti che, nelle ipotesi poste, la connessione ipersuperficiale di τ può anche essere infinita.

Ciò premesso, diremo che σ, A, B appartengono all'insieme (α) se σ soddisfa alle condizioni ora indicate e se A, B sono sommabili su σ.

Converrà inoltre studiare un altro caso, più comune nelle applicazioni, considerando un secondo insieme, (β), contenuto in (α).

Anzitutto, supporremo $A(M)$ continua in tutti i punti di σ, $B(M)$ continua in tutti i punti di $\sigma - \chi$ e tale che l'integrale

$$J_1(N) = \int_\sigma |B(M)| MN^{-(m-2)} d\sigma, \qquad \text{per } m > 2,$$

$$J_1(N) = \int_\sigma |B(M) \log MN| d\sigma, \qquad \text{per } m = 2, \tag{83}$$

considerato per N variable su tutto σ, sia ivi funzione continua.

Supporremo inoltre che gli integrali

$$J_2(R) = \int_\sigma MR^{-(m-2)} d\sigma, \quad J_3(R) = \int_\sigma \left|\frac{\partial}{\partial n} MR^{-(m-2)}\right| d\sigma, \qquad \text{per } m > 2,$$

$$J_2(R) = \int_\sigma |\log MR| d\sigma, \quad J_3(R) = \int_\sigma \left|\frac{\partial}{\partial n} \log MR\right| d\sigma, \qquad \text{per } m = 2, \tag{84}$$

si mantengano limitati al variare del punto R in tutto lo spazio S_m.

Il significato geometrico della limitatezza degli integrali $J_3(R)$ è evi-

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[11] E. E. Levi, "Sulle equazioni lineari totalmente ellittiche alle derivate parziali," *Rendiconti del circolo Matematico di Palermo*, vol. 24 (1907), pp. 275-317. Vedasi anche, dello stesso Autore, "I problemi dei valori al contorno per le equazioni lineari totalmente ellittiche alle derivate parziali," *Memorie Societa Italiana dei XL*, vol. 16 (1910), pp. 3-111. La funzione $V(M, P)$ data dalle (20), (21) è quella ora generalmente considerata (G. Ascoli, *loc. cit.*[2], p. 72) e risulta leggermente diversa da quella introdotta dal Levi.

dente quando si ricordi che, detto $d\omega_{M,R}$ l'angolo secondo il quale è visto dal punto R l'elemento $d\sigma$ relativo al punto M, risulta, come è noto,

$$J_3(R) = \int_\sigma |\, d\omega_{M,R}\,| .$$

Ciò premesso, *indicheremo con* Γ_α *la totalità degli integrali della* (1) *soddisfacenti alle seguenti condizioni*:

a) $u(P)$ *è dato nell'interno di* τ *dalla formula* (16) *dove* σ, A, B *appartengono all'insieme* α *e le funzioni* A, B *soddisfano all'equazione* (15);

b) *risulta, in quasi tutti i punti* M *di* σ,

$$\lim_{P\to M} u(P) = A(M),$$

(85)

$$\lim_{P\to M} \frac{\partial u(P)}{\partial \nu} = B(M),$$

dove $P \to M$ *lungo la conormale* ν *a* σ, *condotta dal punto* M *stesso.*

Indicheremo poi con Γ_β *la totalità degli integrali della* (1) *che si ottengono da* Γ_α *imponendo, come ulteriori condizioni, che* σ, A, B, *appartengano all'insieme* (β), *che la prima delle* (85) *valga in tutti i punti di* σ, *comunque* $P \to M$, *e, infine, che la seconda delle* (85) *valga in tutti i punti di* $\sigma - \chi$ *per* $P \to M$ *lungo la conormale* ν.

3. Dimostriamo ora il seguente teorema fondamentale.

I. *Se le funzioni* $A(M)$, $B(M)$ *soddisfano all'equazione* (15) *in tutti i punto* Q *del dominio* τ' *esterni a* τ, *la funzione* $u(P)$ *definita entro* τ *dalla* (16) *appartiene all'insieme* Γ_α *o* Γ_β *a seconda che* σ, A, B *appartengano all'insieme* (α) *o* (β).

Dimostrazione. Supporremo $m > 2$, la dimostrazione per $m = 2$ essendo del tutto analoga.

Siccome la funzione f è di classe H in τ, la $u(P)$ definita dalla (16) soddisfa all'equazione (1), qualunque siano le funzioni $A(M)$, $B(M)$ sommabili su σ.

a) Ammettiamo dapprima che σ, A, B appartengano all'insieme (β). Sia M_0 un punto qualsiasi di σ. Siccome risulta, per le (13), (14),

$$k_m \equiv \int_\sigma \{\frac{\partial F(M,P)}{\partial \nu} - L(M)F(M,P)\}d\sigma - \int_\tau c(M)F(M,P)d\tau,$$

$$0 \equiv \int_\sigma \{\frac{\partial F(M,Q)}{\partial \nu} - L(M)F(M,Q)\}d\sigma - \int_\tau c(M)F(M,Q)d\tau,$$

si ricava dalle (15), (16)

$$(86)\quad k_m(u(P)-A(M_0))=\int_\sigma\{(A(M)-A(M_0))[\frac{\partial F(M,P)}{\partial\nu}-\frac{\partial F(M,Q}{\partial\nu}$$
$$-L(M)(F(M,P)-F(M,Q))]-B(M)(F(M,P)-F(M,Q))\}d\sigma$$
$$-\int_\tau(f(M)-A(M_0)c(M))(F(M,P)-F(M,Q))d\tau$$

e, per essere $F(M,N)=O(MN^{-(m-2)})$, risulta

$$(87)\quad \lim_{(P,Q)\to M_0}\int_\tau(f(M)-A(M_0)c(M))(F(M,P)-F(M,Q))d\tau=0.$$

Dimostriamo ora che si ha

$$(88)\quad \lim_{(P,Q)\to M_0}\int_\sigma B(M)(F(M,P)-F(M,Q))d\sigma=0.$$

Sia K una costante positiva tale che risulti

$$(89)\quad |F(M,R)|\leq KMR^{-(m-2)}$$

qualunque siani i punti M, R di τ'.

Prefissato $\epsilon>0$ si determini poi, in corrispondenza, $\delta>0$ in modo che, detta σ_δ la parte di σ contenuta nell'ipersfera (M_0,δ), di centro M_0 e raggio δ, risulti

$$(90)\quad \int_{\sigma\delta}|B(M)|MM_0^{-(m-2)}d\sigma_\delta\leq\epsilon/K.$$

Si ha poi, se M_1 è un punto di σ_δ, per la continuità dell'integrale $J_1(N)$ definito dalla (83),

$$\lim_{M_1\to M_0}\int_{\sigma\delta}|B(M)|MM_1^{-(m-2)}d\sigma_\delta=\int_{\sigma\delta}|B(M)|MM_0^{-(m-2)}d\sigma_\delta$$

e quindi si può determinare $\eta\leq\delta$ in modo che, per $M_0M_1\leq\eta$, risulti, per le (89), (90),

$$\int_{\sigma\delta}|B(M)|MM_1^{-(m-2)}d\sigma_\delta<2\epsilon/K$$

e, a maggior ragione,

$$(91)\quad \int_{\sigma\eta}|B(M)|MM_1^{-(m-2)}d\sigma_\eta<2\epsilon/K.$$

Sia ora R un punto qualsiasi di τ', M_R un punto di σ_η situato alla minima distanza da R. Siccome è

$$MM_R\leq MR+RM_R\leq 2MR$$

risulta, per le (89), (91),

$$(92)\qquad \int_{\sigma\eta} |B(M)F(M,R)|\,d\sigma_\eta \leqq K\int_{\sigma\eta} |B(M)|\,MR^{-(m-2)}d\sigma_\eta$$
$$\leq 2^{m-2}K\int_{\sigma\eta} |B(M)|\,MM_R^{-(m-2)}d\sigma_\eta < 2^{m-1}\epsilon.$$

Si ha poi

$$\int_\sigma B(M)(F(M,P)-F(M,Q))d\sigma\,| \leq \int_{\sigma\eta} |B(M)|\,(|F(M,P)|$$
$$+|F(M,Q)|)d\sigma_\eta$$
$$+\,|\int_{\sigma-\sigma\eta} B(M)(F(M,P)-F(M,Q))d\sigma\,|$$

e il terzo integrale a secondo membro è infinitesimo per $(P,Q)\to M_0$. Dalla (92) segue allora la (88).

Proviamo ora che risulta

$$(93)\qquad \lim_{(P,Q)\to M_0}\int_\sigma (A(M)-A(M_0))\,[\frac{\partial F(M,Q)}{\partial \nu}-\frac{\partial F(M,Q)}{\partial \nu}$$
$$-L(M)(F(M,P)-F(M,Q))]d\sigma = 0.$$

Posto $\rho = MR$ e indicate con $(\zeta_1,\cdots,\zeta_m)$ le coordinate del punto R, si ha, per la (24),

$$(94)\qquad \frac{\partial F(M,R)}{\partial \nu} = -(m-2)Z(M,R)^{-m/2}\Sigma(x_k-\zeta_k)\cos(nx_k)+\omega(M,R)$$

con

$$(95)\qquad \omega(M,R) = O(MR^{-(m-2)}).$$

Siccome è

$$\Sigma(x_k-\zeta_k)\cos(nx_k) = \rho(\partial\rho/\partial n),$$

si ricava perciò dalle (92), (95),

$$(96)\qquad \frac{\partial F(M,R)}{\partial \nu} = O(|\frac{\partial \rho^{-(m-2)}}{\partial n}|+\rho^{-(m-2)})$$

e quindi, per la limitatezza degli integrali $J_2(R)$, $J_3(R)$ definiti dalla (84) e per le (89), (96),

$$(97)\qquad \int_\sigma (|\frac{\partial F(M,R)}{\partial \nu}|+|L(M)F(M,R)|)d\sigma \leq H$$

con H costante positiva.

Prefissato $\epsilon > 0$ si determini ora, in corrispondenza $\gamma > 0$ in modo che, per $M_0M \leq \gamma$, risulti $|A(M) - A(M_0)| < \epsilon/H$.

Ne segue, per la (97),

$$(98) \qquad \left| \int_\sigma (A(M) - A(M_0)) \left[\frac{\partial F(M,P)}{\partial \nu} - \frac{\partial F(M,Q)}{\partial \nu} - L(M)(F(M,P) - F(M,Q)) \right] d\sigma \right| < \left| \int_{\sigma - \sigma\gamma} (A(M) - A(M_0)) \left[\frac{\partial F(M,P)}{\partial \nu} - \frac{\partial F(M,Q)}{\partial \nu} - L(M)(F(M,P) - F(M,Q)) \right] d\sigma \right| + 2\epsilon$$

e quindi, essendo l'integrale a secondo membro infinitesimo per $(P,Q) \to M_0$, la (93) è dimostrata.

Dalle (87), (88), (93) risulta poi

$$(99) \qquad \lim_{P \to M_0} u(P) = A(M_0).$$

Proviamo ora che, se M_0 è un punto di $\sigma - \chi$, detta λ la conormale in M_0 a σ, si ha

$$(100) \qquad \lim_{P \to M_0} \frac{\partial u(P)}{\partial \lambda} = B(M_0),$$

dove $P \to M_0$ lungo la conormale in M_0.

Portiamo, con un cambiamento di coordinate, in M_0 l'origine Ω delle coordinate e assumiamo come asse x_m la normale a σ in M_0 orientata verso l'interno di τ.

Si ha allora, per la (26), applicando le formule (13), (14) alla funzione $x_m/a_{mm}(M_0)$ e supposto P sulla conormale in M_0,

$$k_m\lambda = \int_\sigma \left\{ \frac{x_m}{a_{mm}(M_0)} \left(\frac{\partial F(M,P)}{\partial \nu} - L(M)F(M,P) \right) - \frac{a_{mm}(M)}{a_{mm}(M_0)} F(M,P) \right\} d\sigma - \int_\tau \frac{b_m(M) + x_m c(M)}{a_{mm}(M_0)} F(M,P)\, d\tau$$

$$0 \equiv \int_\sigma \left\{ \frac{x_m}{a_{mm}(M_0)} \left(\frac{\partial F(M,Q)}{\partial \nu} - L(M)F(M,Q) \right) - \frac{a_{mm}(M)}{a_{mm}(M_0)} F(M,Q) \right\} d\sigma - \int_\tau \frac{b_m(M) + x_m c(M)}{a_{mm}(M_0)} F(M,Q)\, d\tau.$$

Ne segue

$$k_m(u(P) - A(M_0) - \lambda B(M_0)) = \int_\sigma \Big\{ \Big(A(M) - A(M_0) - \frac{x_m B(M_0)}{a_{mm}(M_0)} \Big) \Big[\frac{\partial F(M,P)}{\partial \nu} + \frac{\partial F(M,Q)}{\partial \nu} - L(M)(F(M,P) + F(M,Q)) \Big] - \Big(B(M) - \frac{a_{mm}(M) B(M_0)}{a_{mm}(M_0)} \Big)(F(M,P) + F(M,Q)) \Big\} d\sigma - \int_\tau \Big(f(M) - c(M) A(M_0) - \frac{(b_m(M) + x_m c(M)) B(M_0)}{a_{mm}(M_0)} \Big) (F(M,P) + F(M,Q)) d\tau$$

e quindi, assunto come punto Q il simmetrico di P rispetto a M_0,

$$(101) \quad k_m \Big(\frac{\partial u(P)}{\partial \lambda} - B(M_0) \Big) = \int_\sigma \Big\{ \Big(A(M) - A(M_0) - \frac{x_m B(M_0)}{a_{mm}(M_0)} \Big) \Big[\frac{\partial^2 F(M,P)}{\partial \nu \partial \lambda} + \frac{\partial^2 F(M,Q)}{\partial \nu \partial \lambda} - L(M) \Big(\frac{\partial F(M,P)}{\partial \lambda} + \frac{\partial F(M,Q)}{\partial \lambda} \Big) \Big] - \Big(B(M) - \frac{a_{mm}(M) B(M_0)}{a_{mm}(M_0)} \Big) \Big(\frac{\partial F(M,P)}{\partial \lambda} + \frac{\partial F(M,Q)}{\partial \lambda} \Big) \Big\} d\sigma - \int_\tau \Big(f(M) - c(M) A(M_0) - \frac{(b_m(M) + x_m c(M)) B(M_0)}{a_{mm}(M_0)} \Big) \Big(\frac{\partial F(M,P)}{\partial \lambda} + \frac{\partial F(M,Q)}{\partial \lambda} \Big) d\tau.$$

Osserviamo ora che risulta, uniformemente, al variare di M in un insieme chiuso non contenente M_0,

$$(102) \quad \lim_{\lambda \to 0} \Big(\frac{\partial F(M,P)}{\partial \lambda} + \frac{\partial F(M,Q)}{\partial \lambda} \Big) = \lim_{\lambda \to 0} \Big(\frac{\partial^2 F(M,P)}{\partial \nu \partial \lambda} + \frac{\partial^2 F(M,Q)}{\partial \nu \partial \lambda} \Big) = 0.$$

Per il lemma I si possono determinare due numeri positivi h, k in modo che, per $M_0 M \leq h$, $0 < \lambda \leq h$, risulti

$$\Big| \frac{\partial^2 F(M,P)}{\partial \nu \partial \lambda} + \frac{\partial^2 F(M,Q)}{\partial \nu \partial \lambda} \Big| + (1 + |L(M)|) \Big| \frac{\partial F(M,P)}{\partial \lambda} + \frac{\partial F(M,Q)}{\partial \lambda} \Big| \leq k \Big(\frac{\lambda}{(s^2 + \lambda^2)^{m/2}} + \frac{1}{s^{m-2}} \Big)$$

e quindi

$$(103)\qquad \int_{\sigma_h}\Big\{\Big|\frac{\partial^2F(M,P)}{\partial\nu\partial\lambda}+\frac{\partial^2F(M,Q)}{\partial\nu\partial\lambda}\Big|+(1+|L(M)|)\Big|\frac{\partial F(M,P)}{\partial\lambda}+\frac{\partial F(M,Q)}{\partial\lambda}\Big|\Big\}d\sigma_h$$
$$\leq k'\int_0^h s^{m-2}\Big(\frac{\lambda}{(s^2+\lambda^2)^{m/2}}+\frac{1}{s^{m-2}}\Big)ds<k'\Big(\int_0^\infty\frac{\lambda}{s^2+\lambda^2}\,ds+h\Big)$$
$$=k'(\pi/2+h)=k''$$

con k', k'' costanti positive.

Prefissato $\epsilon>0$, si determini in corrispondenza δ, con $0<\delta\leq h$, in modo che per $M_0M\leq\delta$ risulti

$$(104)\qquad \Big|A(M)-A(M_0)-\frac{x_mB(M_0)}{a_{mm}(M_0)}\Big|<\epsilon/k'',$$
$$\Big|B(M)-B(M_0)\frac{a_{mm}(M)}{a_{mm}(M_0)}\Big|<\epsilon/k''.$$

Dalle (101), (103), (104) segue allora

$$(105)\qquad |k_m(\partial u(P)/\partial\lambda-B(M_0))|\leq\epsilon$$
$$+\Big|\int_{\sigma-\sigma\delta}\Big\{\Big(A(M)-A(M_0)-\frac{x_mB(M_0)}{a_{mm}(M_0)}\Big)\Big[\frac{\partial^2F(M,P)}{\partial\nu\partial\lambda}+\frac{\partial^2F(M,Q)}{\partial\nu\partial\lambda}$$
$$-L(M)\Big(\frac{\partial F(M,P)}{\partial\lambda}+\frac{\partial F(M,Q)}{\partial\lambda}\Big)\Big]-\Big(B(M)-\frac{a_{mm}(M)B(M_0)}{a_{mm}(M_0)}\Big)$$
$$\Big(\frac{\partial F(M,P)}{\partial\lambda}+\frac{\partial F(M,Q)}{\partial\lambda}\Big)\Big\}d\sigma\Big|$$
$$+\Big|\int_\tau\Big(f(M)-c(M)A(M_0)-\frac{(b_m(M)+x_mc(M))B(M_0)}{a_{mm}(M_0)}\Big)$$
$$\Big(\frac{\partial F(M,P)}{\partial\lambda}+\frac{\partial F(M,Q)}{\partial\lambda}\Big)d\tau\Big|.$$

Per la (102) e per le limitazioni, valide in tutti i punti M di τ,

$$\Big|\frac{\partial F(M,P)}{\partial\lambda}\Big|\leq KMP^{-(m-1)},$$
$$\Big|\frac{\partial F(M,Q)}{\partial\lambda}\Big|\leq KMQ^{-(m-1)},$$

(dove K è una costante positiva) gli integrali a secondo membro della (105) sono infinitesimo con λ e quindi dalla (105) segue la (100).

b) Supponiamo ora che σ, A, B appartengano all'insieme (α). In tal caso ricordiamo che risulta quasi ovunque su $\sigma - \chi$ e quindi su σ, per la sommabilità di A, B,

$$(106)\quad \lim_{\delta\to 0} (1/\delta^{m-1} \int_{\sigma\delta} \{|A(M) - A(M_0)| + |B(M) - B(M_0)|\} d\sigma_\delta = 0.$$

Sia M_0 un punto di $\sigma - \chi$ in cui vale la (106). Sarà perciò anche, in M_0,

$$(107)\quad \lim_{\delta\to 0} \int_{\sigma\delta} \left\{ \left| A(M) - A(M_0) - \frac{x_m B(M_0)}{a_{mm}(M_0)} \right| + \left| B(M) - \frac{a_{mm}(M) B(M_0)}{a_{mm}(M_0)} \right| \right\} d\sigma_\delta = 0.$$

Prefissato $\epsilon > 0$ e posto

$$\psi(t) = \int_{\sigma_t} \{|A(M) - A(M_0)| + |B(M) - B(M_0)|\} d\sigma_t,$$

si determini η, con $0 < \eta \leq \epsilon$, in modo che, per $0 \leq t \leq \eta$, risulti

$$0 \leq \psi(t) \leq \epsilon\, t^{m-1}.$$

Si ricava allora, per il lemma I, indicata con k_1 una costante positiva indipendente da ϵ e supposti P e Q sulla conormale in M_0 (sicchè risulterà $|F(M,P)| + |F(M,Q)| = 0(t^{-(m-2)})$),

$$(108)\quad \Big| \int_{\sigma\eta} \Big\{ (A(M) - A(M_0)) \Big[\frac{\partial F(M,P)}{\partial \nu} - \frac{\partial F(M,Q)}{\partial \nu} - L(M) \wedge F(M,P) - F(M,Q)) - B(M)[F(M,P) - (F(M,Q]\} d\sigma_\eta \Big|$$

$$\leq k_1 \Big[\int_0^\eta \left(\frac{\lambda}{(t^2+\lambda^2)^{m/2}} + \frac{1}{t^{m-2}} \right) \psi'(t \mid dt + \eta \mid B(M_0)|] = k_1 [\psi(\eta) \left(\frac{\lambda}{(\eta^2+\lambda^2)^{m/2}} + \frac{1}{\eta^{m-2}} \right) + \int_0^\eta \psi(t) \left(\frac{mt\lambda}{(t^2+\lambda^2)^{m+2/2}} + \frac{m-2}{t^{m-1}} \right) dt$$

$$+ \eta \mid B(M_0)|] < \epsilon k_1 \Big[\frac{\eta^{m-1}\lambda}{(\eta^2+\lambda^2)^{m/2}} + \eta + \int_0^\eta \frac{mt^m\lambda}{(t^2+\lambda^2)^{m+2/2}} + m - 2) dt + |B(M_0)| \Big] < \epsilon k_1 (1 + \eta + m\pi/2) + (m-2)\eta + |B(M_0)|).$$

Dalla (108) segue poi, per la (86),

$$\lim_{P\to M_0} u(P) = A(M_0),$$

dove $P \to M_0$ lungo la conormale λ.

Poniamo ora

$$\omega(t) = \int_{\sigma_t} \left\{ \left| A(M) - A(M_0) - \frac{x_m B(M_0)}{a_{mm}(M_0)} \right| + \left| B(M) - \frac{a_{mm}(M) B(M_0)}{a_{mm}(M_0)} \right| \right\} d\sigma_t$$

e determiniamo $\zeta > 0$ in modo che, per $0 \leq t \leq \zeta$, risulti, per la (107),

$$0 \leq \omega(t) \leq \epsilon t^{m-1}.$$

Si ricava allora, per il lemma I, indicata con h_1 una costante positiva indipendente da ϵ e supposti P, Q sulla conormale in M_0 ed equidistanti da tale punto,

$$\left| \int_{\sigma_\xi} \left\{ \left(A(M) - A(M_0) - \frac{x_m B(M_0)}{a_{mm}(M_0)} \right) \left[\frac{\partial^2 F(M,P)}{\partial \nu \partial \lambda} + \frac{\partial^2 F(M,Q)}{\partial \nu \partial \lambda} - L(M) \left(\frac{\partial F(M,P)}{\partial \lambda} + \frac{\partial F(M,Q)}{\partial \lambda} \right) \right] - \left(B(M) - \frac{a_{mm}(M) B(M_0)}{a_{mm}(M_0)} \right) \left(\frac{\partial F(M,P)}{\partial \lambda} + \frac{\partial F(M,Q)}{\partial \lambda} \right) \right\} d\sigma_\xi \right|$$

$$\leq h_1 \int_0^\xi \left\{ \frac{\lambda}{(t^2+\lambda^2)^{m/2}} + \frac{1}{t^{m-2}} \right\} \omega'(t)\, dt$$

e quindi, ragionando come precedentemente, si deduce

$$\lim_{P \to M_0} \frac{\partial u(P)}{\partial \lambda} = B(M_0)$$

dove $P \to M_0$ lungo la sonormale λ in tale punto.

4. Dal teorema I segue che l'equazione (15) *caratterizza* tutte le coppie A, B di funzioni coincidenti rispettivamente con i valori assunti su σ da un integrale u della (1), appartenente all'insieme Γ_α o Γ_β, e dalla sua derivata conormale $\partial u/\partial \nu$.

Essa può inoltre essere utilizzata per risolvere non solo i problemi di Dirichlet e di Neumann ma anche il problema misto (che li comprende come casi particolari) ed il problema di Cauchy (in grande).

Supponiamo infatti assegnata su σ una combinazione lineare delle funzioni u e $\partial u/\partial \nu$:

$$hu + k(\partial u/\partial \nu) = C, \tag{109}$$

con h, k, C funzioni assegnate, le prime due non contemporaneamente nulle, escluso al più un insieme γ di misura ipersuperficiale nulla. Ammettiamo inoltre che $\sigma - \gamma$ si possa dividere in due insiemi, σ_1 e σ_2, in modo che, in σ_1, la funzione h/k sia limitata e la C/k sommabile, in σ_2 sia limitata la funzione k/h e la C/h sommabile.

Poniamo poi nella (109) A, B in luogo di u, $\partial u/\partial \nu$ e supponiamo che la funzione $u(P)$ appartenga all'insieme Γ_α (in modo simile si può studiare il caso in cui $u(P)$ appartiene all'insieme Γ_β).

Sarà, su σ_1,

$$B = -(h/k)A + C/k \tag{110}$$

e, su σ_2,

$$A = -(k/h)B + C/h. \tag{111}$$

Introdotta la successione $\{v_r\}$ data dalla (17), risulta, per le (18), (110), (111)

$$\begin{aligned} (112)\quad & \int_{\sigma_0} A\{(\partial v_r/\partial \nu) - v_r(L - (h/k))\}d\sigma_1 \\ & - \int_{\sigma_2} B\left\{\frac{k}{h}\frac{\partial v_r}{\partial \nu} + v_r(1 - L(k/h))\right\}d\sigma_2 \\ & = \int_{\sigma_1} (C/k)v_r d\sigma_1 - \int_{\sigma_2} (C/h)((\partial v_r/\partial \nu) - Lv_r d\sigma_2 + \int_\tau f v_r d\tau. \end{aligned}$$

Se definiamo perciò un vettore $\boldsymbol{G}$ di componenti A in σ_1, B in σ_2 e una successione di vettori $\{\boldsymbol{\omega}_r\}$, di componenti $\{(\partial v_r/\partial \nu) - v_r(L - (h/k))\}$ in σ_1, $-\{\frac{k}{h}\frac{\partial v_r}{\partial \nu} + v_r(1 - L(k/h))\}$ in σ_2, e indichiamo con c_r il secondo membro della (112), ricaviamo il sistema di Fischer-Riesz

$$(\boldsymbol{G}, \boldsymbol{\omega}_r) = c_r \tag{113}$$

cioè risultano noti coefficienti di Fourier dell'incognito vettore $\boldsymbol{G}$ rispetto alla successione $\{\boldsymbol{\omega}_r\}$.

Viceversa, sia $\boldsymbol{G}$ una soluzione del sistema (113). Si conoscono allora le funzioni A su σ_1, B su σ_2; definite poi, su σ_1 e σ_2 rispettivamente, le B, A mediante le (110) e (111), dalla (112), eliminando C con le (110), (111), segue che la coppia A, B soddisfa al sistema (18) e quindi all'equivalente equazione (15). Per il teorema I la funzione $u(P)$ data dalla (16) è un integrale della (1) tale che risulti, su σ, $u = A$, $\partial u/\partial \nu = B$. Per le (110), (111) è allora verificata su σ la (109).

Inoltre, se per il problema considerato vale, nell'insieme Γ_a, il teorema di unicità, la sucessione $\{\omega_r\}$ risulta chiusa rispetto alla totalità dei vettori G di componenti sommabili; se tale teorema non vale, le autosoluzioni sono *caratterizzate* dall'essere i corrispondenti vettori G ortogonali su σ alla successione $\{\omega_r\}$.

Infine, *se il problema non ammette soluzione, nemmeno il sistema* (113) *ammette soluzione.*

Dedotta poi dalla successione $\{\omega_r\}$ una equivalente successione $\{\zeta_r\}$ ortogonale e normale, il sistema (113) *equivale* al sistema

$$(G, \zeta_r) = d_r \tag{114}$$

dove le d_r sono costanti note. Perciò *condizione necessaria e sufficiente affinchè il problema ammetta soluzione, il vettore G risultando di norma integrale finita, è che converga la serie Σd^2_r.*

In modo analogo si ragiona per il problema di Cauchy, nel quale supporremo assegnati su una parte, σ_1, di σ i valori di u, e $\partial u/\partial v$. Posto $\sigma_2 = \sigma - \sigma_1$ e, al solito, su σ, $u = A$, $\partial u/\partial v = B$, si ricava dal sistema (18) il sistema

$$\int_{\sigma_2} \{A((\partial v_r/\partial v) - Lv_r) - Bv_r\} d\sigma_2 = -\int_{\sigma_1} \{A((\partial v_r/\partial v) - Lv_r) - Bv_r\} d\sigma_1 + \int_\tau f v_r d\tau = c_r, \tag{115}$$

con c_r costanti note. Considerate poi le A, B, in σ_2, come componenti di un vettore G, le $((\partial v_r/\partial v) - Lv_r)$, $-v_r$ come componenti di un vettore ω_r, il sistema (115) diventa

$$\{G, \omega_r\} = c_r$$

e si possono ripetere le considerazioni svolte a proposito dei sistemi (113), (114).

II. Secondo procedimento di integrazione.

1. Dimostriamo due lemmi dei quali faremo uso in seguito.

LEMMA II. *Sia $U(M, R)$ una funzione definita nel dominio τ' contenente τ nel suo interno, la quale soddisfi, uniformemente al variare di M, R in τ', alle limitazioni*

$$U(M,R) = O(MR^{-(m-2)}),$$

$$\frac{\partial U(M,R)}{\partial x_i} = O(MR^{-(m-1)}),$$

$$\frac{\partial^2 U(M,R)}{\partial x_i \partial x_k} = O(MR^{-m}).$$

dove $(x_1, \cdots, x_m)$ *sono le coordinate di* M.

Indicato con d *il diametro del dominio* τ' *e posto, per* r *intero positivo,* $M, N(y_1, \cdots, y_m)$ *in* τ',

$$\phi_r(N,M) = 1/d^m (r/\pi)^{m/2} \{(1 - ((x_1 - y_1)^2/d^2)) \cdots \times (1 - ((x_m - y_m)^2/d^2))\}^2,$$

il polinomio di Stieltjes relativo alla $U(M,R)$:

$$U_r(M,R) = \int_{\sigma'} U(N,R)\phi_r(N,M)\,d\tau'$$

soddisfa, uniformemente rispetto a r *e al variare di* M, R *nel dominio* τ *interno a* τ', *alle limitazioni*

$$U_r(M,R) = O(MR^{-(m-2)}),$$

$$\frac{\partial U_r(M,R)}{\partial x_i} = O(MR^{-(m-1)}),$$

$$\frac{\partial^2 U_r(M,R)}{\partial x_i \partial x_k} = O(MR^{-m}).$$

Dimostrazione. a) Poniamo

(116) $$\delta = MR, \qquad \rho = NR, \qquad t = MN$$

e sia τ'_R il dominio ipersferico di centro R e raggio $\delta/2$, σ'_R l'ipersfera $(R, \delta/2)$, contorno di τ'_R.

Si ricava allora, indicata con K una opportuna costante positiva e con σ_m la misura dell'ipersfera unitaria in S_m,

(117) $$|U_r(M,R)| \leqq K r^{m/2} \int_{\tau'} \frac{e^{-r(t^2/d^2)}}{\rho^{m-2}}\,d\tau' = K\{r^{m/2} \int_{\tau'_R} \frac{e^{-r(t^2/d^2)}}{\rho^{m-2}}\,d\tau'_R$$

$$+ r^{m/2} \int_{\tau' - \tau'_R} \frac{e^{-r(t^2/d^2)}}{\rho^{m-2}}\,d\tau'\} < K\{r^{m/2} e^{-r(\delta^2/4d^2)} \int_{\tau'_R} \frac{d\tau'}{\rho^{m-1}}$$

$$+ r^{m/2}(2/\delta)^{m-2} \int_{-\infty}^{\infty} e^{-r((x_1-y_1)^2/d^2)}\,dy_1 \cdots \int_{-\infty}^{\infty} e^{-r((x_m-y_m)^2/d^2)}\,dy_m\}$$

$$= K\left\{ r^{m/2}\sigma_m(\delta^2/2)\,e^{-r(\delta^2/4d^2)} + 2^{m-2}\frac{d^m \pi^{m/2}}{\delta^{m-2}} \right\} = O(\delta^{-(m-2)}),$$

uniformemente rispetto a r e al variare di M, R in τ.

b) Siccome è $|U| \leq K\delta^{-(m-2)}$, con K costante positiva, risulta, supposto che τ' abbia contorno σ' di classe 1,

$$(118)\quad \frac{\partial U_r(M,R)}{\partial x_i} = \int_{\tau'} U(N,R)\frac{\partial \phi_r(N,M)}{\partial x_i}\,d\tau'$$

$$= -\int_{\tau'} U(N,R)\frac{\partial \phi_r(N,M)}{\partial y_i}\,d\tau'$$

$$= -\int_{\tau'} \frac{\partial}{\partial y_i}\left(U(N,R)\phi_r(N,M)\right)d\tau' + \int_{\tau'} \frac{\partial U(N,R)}{\partial y_i}\phi_r(N,M)\,d\tau'$$

$$= \int_{\sigma'} U(N,R)\phi_r(N,M)\cos(n'y_i)\,d\sigma' + \int_{\tau'} \frac{\partial U(N,R)}{\partial y_i}\phi_r(N,M)\,d\tau'$$

dove n' è la normale in N a σ', orientata verso l'interno di τ'.

Indicata con $h > 0$ la distanza del dominio τ da $\delta\sigma'$, con K' une costante positiva e con μ' la misura di σ', si ha poi, essendo M ed R punti di τ,

$$(119)\quad \left|\int_{\sigma'} U(N,R)\phi_r(N,M)\cos(n'y_i)\,d\sigma'\right| \leq \frac{K'\mu'}{h^{m-2}}\,r^{m/2}e^{-r(h^2/d^2)} = O(1),$$

$$(120)\quad \left|\int_{\tau'} \frac{\partial U(N,R)}{\partial y_i}\phi_r(N,M)\,d\tau'\right| \leq K'\Big\{r^{m/2}e^{-r(h^2/4d^2)}\int_{\tau'_R}\frac{d\tau'_R}{\rho^{m-1}}$$

$$+ r^{m/2}(2/\delta)^{m-1}\int_{-\infty}^{\infty} e^{-r((x_1-y_1)2/d^2)}dy\cdots\int_{-\infty}^{\infty} e^{-r((x_m-y_m)2d/^2)}dy_m\Big\} = O(\delta^{-(m-1)}),$$

Dalle (118), (119), (120) segue allora

$$(121)\quad \frac{\partial U_r(M,R)}{\partial x_i} = O(MR^{-(m-1)}),$$

uniformemente rispetto a r e al variare di M, R in τ.

c) Risulta, per la (118), se n'_R è la normale a σ'_R orientata verso l'esterno di τ'_R,

$$(122)\quad \frac{\partial^2 U(M,R)}{\partial x_i \partial x_k}\int_{\sigma'} U(N,R)\frac{\partial \phi_r(N,M)}{\partial x_k}\cos(n'y_i)\,d\sigma'$$

$$-\int_{\tau'-\tau'_R}\frac{\partial U(N,R)}{\partial y_i}\,\frac{\partial \phi_r(N,M)}{\partial y_k}\,d\tau'$$

$$+\int_{\tau'_R}\frac{\partial U(N,R)}{\partial y_i}\,\frac{\partial \phi_r(N,M)}{\partial x_k}\,d\tau'_R$$

$$-\int_{\sigma'} U(N,R)\frac{\partial \phi_r(N,M)}{\partial x_k}\cos(n'y_i)\,d\sigma'$$

$$-\int_{\tau'-\tau'_R}\frac{\partial}{\partial y_k}\left(\frac{\partial U(N,R)}{\partial y_i}\,\phi_r(N,M)\right)d\tau'$$

$$+\int_{\tau'-\tau'_R}\frac{\partial^2 U(N,R)}{\partial y_i\partial y_k}\phi_r(N,M)\,d\tau'+\int_{\tau'_R}\frac{\partial U(N,R)}{\partial y_i}\,\frac{\partial\phi_2(N,M)}{\partial x_k}\,d\tau'_R$$

$$=\int_{\sigma'}\{U(N,R)\,\frac{\partial\phi_r(N,M)}{\partial x_k}\cos(n'y_i)$$

$$+\frac{\partial U(N,R)}{\partial y_i}\,\phi_r(N,M)\cos(n'y_k)\}\,d\sigma'$$

$$+\int_{\sigma'_R}\frac{\partial U(N,R)}{\partial y_i}\,\phi_r(N,M)\cos(n'_R y_k)\,d\sigma'$$

$$+\int_{\tau'-\tau'_R}\frac{\partial^2 U(N,R)}{\partial y_i\partial y_k}\,\phi_r(N,M)\,d\tau'$$

$$+\int_{\tau'_R}\frac{\partial U(N,R)}{\partial y_i}\,\frac{\partial\phi_2(N,M)}{\partial x_k}\,d\tau'_R$$

e si ha, indicata con K'' una costante positiva,

$$(123)\quad \Big|\int_{\sigma'}\{U(N,R)\,\frac{\partial\phi_r(N,M)}{\partial x_k}\cos(n'y_i)\cdot$$

$$+\frac{\partial U(N,R)}{\partial y_i}\,\phi_r(N,M)\cos(n'y_k)\}\,d\sigma'\Big|$$

$$\leq K''\mu'\{(1/h^{m-2})\,r^{m+2/2}e^{-r(\lambda^2/d^2)}+(1/h^{m-1})\,r^{m/2}e^{-r(\lambda^2/d^2)}\}=O(1),$$

$$(124)\quad \Big|\int_{\sigma'_R}\frac{\partial U(N,R)}{\partial y_i}\,\phi_r(N,M)\cos(n'_R y_k(d\sigma'_R\,|\,K''(r^{m/2}/\delta^{m-1})\,e^{-r(\delta^2/4d^2)}\delta^{m-1}$$

$$=O(\delta^{-m}),$$

$$(125)\quad \Big|\int_{\tau'-\tau'_R}\frac{\partial^2 U(N,R)}{\partial y_i\partial y_k}\,\phi_r(N,M)\,d\tau'\Big|\leq K''r^{m/2}(2/\delta)^m$$

$$\int_{\tau'}e^{-r(t^2/d^2)}\,d\tau'=O(\delta^{-m}),$$

$$(126)\quad \Big|\int_{\tau'_R}\frac{\partial U(N,R)}{\partial y_i}\,\frac{\partial\phi_r(N,M)}{\partial x_k}\,d\tau'_R\Big|\leq K''r^{(m+2/2)}$$

$$\int_{\tau'_R}(|\,x_k-y_k\,|/\rho^{m-1})\,e^{-r(t^2/d^2)}\,d\tau'_R$$

$$<K''r^{(m+2/2)}\,e^{-r(\delta^2/4d^2)}3/2\delta^2\sigma_m=O(\delta^{-m}).$$

Dalle (122), (123), (124), (125), (126) segue allora, uniformamente rispetto ad r e al variare di M, R in τ,

$$(127)\qquad \frac{\partial^2 U_r(M,R)}{\partial x_i\partial x_k}=O(MR^{-m}).$$

Osservazione. Dalla dimostrazione data in a) si deduce che *se risulta, uniformemente, al variare di* M, R *in* τ',

$$U(M,R) = O(MR^{-k}),$$

con $k < m$, si ha anche, uniformemente rispetto a r e al variare di M, R in τ,

$$U_r(M,R) = O(MR^{-k}).$$

LEMMA III. *Se la funzione $u(M)$ è sommabile in τ, lo è anche la funzione $u(M)MP^{-k}$, con $k < m$, esclusi al più, in τ, i punti P di un insieme di misura nulla.*

Dimostrazione. Se τ_ρ è il dominio ipersferico limitato dall'ipersfera (P,ρ), si ha quasi ovunque in τ, porto $\rho = PM$,

$$\lim_{\rho\to 0} 1/\rho^m \int_{\tau_\rho} |u(M) - u(P)|\, d\tau_\rho = 0. \tag{128}$$

Sia P un punto in cui vale la (128). Posto allora

$$\psi(\rho) = \int_{\tau_\rho} |u(M) - u(P)|\, d\tau_\rho,$$

si determini $\delta > 0$ in modo che, per $0 \leq \rho \leq \delta$, risulti $0 \leq \psi(\rho) \leq \rho^m$.

Si ricava allora, essendo σ_m la misura dell'ipersfera unitaria in S_m,

$$\begin{aligned}\int_{\tau_\delta} (|u(M)|/\rho^k)\, d\tau_\delta &\leq u(P) \int_{\tau_\delta} d\tau_\delta/\rho^k + \int_{\tau_\delta} (|u(M) - u(P)|/\rho^k)\, d\tau_\delta \\ &= |u(P)|\, \sigma_m(\delta^{m-k}/m - k) + \int_0^\delta (\psi'(\rho/\rho^k)\, d\rho \\ &= |u(P)|\, \sigma_m(\delta^{m-k}/m - k) + \psi(\delta)/\delta^k + k\int_0^\delta (\psi(\rho)/\rho^{k+1})\, d\rho \\ &\leq |u(P)|\, \sigma_m(\delta^{m-k}/m - k) + \delta^{m-k} + k(\delta^{m-k}/m - k)\end{aligned}$$

da cui la tesi.

2. Indichiamo ora, con il seguente teorema, una notevole proprietà delle funzioni dell'insieme Γ_a.

II. *Se w è una funzione di classe 2 nel dominio τ' e u è una funzione dell'insieme Γ_a, vale la formula di Green*

$$-\int_\tau uE^*(w)\, d\tau = \int_\sigma \{A(\partial w/\partial \nu - Lw) - Bw\}\, d\sigma - \int_\tau fw\, d\tau.$$

Dimostrazione. La funzione w è in τ' un integrale di classe 2 dell'equazione $E^*(v) = E^*(w)$. Possiamo inoltre supporre che il dominio τ' contenente τ nel suo interno abbia contorno di classe 1 e sia interno a un dominio τ'' nel quale è definita la soluzione fondamentale.

Si ricava allora, per R interno a τ',

$$k_m w(R) = \int_{\sigma'} \{w(N)((\partial F(R,N)/\partial \nu') + L(N)F(R,N)) - (\partial w(N)/\partial \nu')F(R,N)\}d\sigma' - \int_{\tau'} E^*(w(N))F(R,N)d\tau',$$

dove ν' è la conormale a σ', orientata verso l'interno di τ', e N è il punto, di σ' o di τ', rispetto alle cui coordinate si effettuano le integrazioni.

Ne segue, per le (15), (16) e oservando che i contorni σ e σ' hanno tra loro distanza positiva,

$$\begin{aligned}
&\int_\sigma \{A(M)(\partial w(M)/\partial \nu) - (B(M) + L(M)A(M)w(M)\}d\sigma \\
&\qquad - \int_\tau f(M)m(M)d\tau \\
&= \int_\sigma \{A(M)\Big[\frac{1}{k_m}\frac{\partial}{\partial \nu}\int_{\sigma'}(w(N)((\partial F(M,N)/\partial \nu') + L(N)F(M,N) \\
&\qquad - (\partial w(N)/\partial \nu')F(M,N))d\sigma' \\
&\quad - \frac{1}{k_m}\frac{\partial}{\partial \nu}\int_{\tau'} E^*(w(N))F(M,N)d\tau'\Big] \\
&\quad - (B(M) + L(M)A(M))[1/k_m \int_{\sigma'}(w(N)((\partial F(M,N)/\partial \nu') \\
&\qquad + L(N)F(M,N)) - (\partial w(N)/\partial \nu')F(M,N))d\sigma' \\
&\quad - 1/k_m \int_{\tau'} E^*(w(N))F(M,N)d\tau']\}d\sigma \\
&\quad - \int_\tau f(M)[1/k_m \int_{\sigma'}(w(N)((\partial F(M,N)/\partial \nu') + L(N)F(M,N) \\
&\qquad (\partial w(N)/\partial \nu')F(M,N))d\sigma' \\
&\quad - 1/k_m \int_{\tau'} E^*(w(N)F(M,N)d\tau']d\tau \\
&= \int_{\sigma'}\{w(N)(\partial/\partial \nu'[1/k_m \int_\sigma (A(M)((\partial F(M,N)/\partial \nu) \\
&\qquad - L(M)F(M,N) - B(M)F(M,N))d\sigma \\
&\quad - 1/k_m \int_\tau f(M)F(M,N)d\tau] \\
&- ((\partial w(N)/\partial \nu') - L(N)w(N))[1/k_m \int_\sigma (A(M)((\partial F(M,N)/\partial \nu) \\
&\qquad - L(M)F(M,N) - B(M)F(M,N))d\sigma
\end{aligned}$$

$$-1/k_m \int_\tau f(M)F(M,N)d\tau]\}d\sigma'$$

$$-\int_{\tau'} E^*(w(N))[1/k_m \int_\sigma (A(M)((\partial F(M,N)/\partial \nu)$$

$$-L(M)F(M,N) - B(M)F(M,N))d\sigma$$

$$-1/k_m \int_\tau f(M)F(M,N)d\tau]d\tau' = -\int_\tau uE^*(w)d\tau$$

ciò che prova la tesi.

3. Consideriamo in τ' la successione di funzioni

$$w_r = x_1^{\alpha_1} \cdots x_m^{\alpha_m} \qquad (\alpha_i = 0, 1, \cdots). \tag{129}$$

Applicando formalmente la formula di Green alle funzioni u, w_r e posto, al solito, su σ, $u = A$, $\partial u/\partial \nu = B$, ricaviamo il sistema di infinite equazioni di Fischer-Riesz

$$-\int_\tau uE^*(w_r)d\tau = \int_\sigma \{A((\partial w_r/\partial \nu) - Lw_r) - Bw_r\}d\sigma - \int_\tau fw_r d\tau \tag{130}$$

per le quali vale il seguente teorema.

III. *Condizione necessaria e sufficiente perchè le funzioni u, A, B sommabili, la prima su τ, le rimanenti su σ, soddisfino al sistema* (130) *è che la stesse funzioni, alterando al più il valore della u in un insieme di misura nulla, soddisfino alle* (15), (16).

Dimostrazione. Che la condizione sia sufficiente segue dal Teorema II.

Dimostriamo che la condizione è necessaria. Per questo cominciamo con l'osservare che, se u, A, B soddisfano al sistema (130), soddisfano anche all'equazione

$$-\int_\sigma uE^*(F_r)d\tau = \int_\sigma \{A((\partial F_r/\partial \nu) - LF_r) - BF_r\}d\sigma - \int_\tau fF_r d\tau \tag{131}$$

essendo F_r un qualsiasi polinomio nelle variabili $(x_1, \cdots, x_m)$.

Assumiano, in particolare, come polinomio F_r il polinomio di Stieltjes relativo alla soluzione fondamentale $F(M,R)$,[12] definito dall'eguaglianza

[12] Se la $F(M,P)$ data dalla (76) non soddisfa, come funzione di P, alla $E(u) = 0$, si può sostituirla con la funzione di Green $G(M,P)$ relativa a un problema al contorno per la (4) in un dominio τ'' contenente τ al sua interno (M. Gevrey, "Détermination et emploi des fonctions de Green dans les problèmes aux limites relatifs aux équations linéaires du type elliptique," *Journal des Mathématiques*, vol. 9 (1930), p. 11).

$$F_r(M,R) = \int_{\tau'} F(N,R)\phi_r(N,M)\,d\tau',$$

essendo $\phi_r(N,M)$ il polinomio considerato nel Lemma II.

Per noti teoremi di Tonelli,[13] il polinomio $F_r(M,R)$ converge uniformemente, per $r \to \infty$, alla $F(M,R)$ in ogni dominio interno a τ' ma con contenente R. In tale dominio anche le derivate $\partial F_r/\partial x_i$, $\partial^2 F_r/\partial x_i \partial x_k$, convergono uniformemente a $\partial F/\partial x_i$, $\partial^2 F/\partial x_i \partial x_k$.

Sia ora R esterno a τ, poniamo cioè $R = Q$.

Risulta allora uniformemente, al variare di M in τ,

$$\lim_{r\to\infty} E^*(F_r(M,Q)) = E^*(F(M,Q)) = 0$$

e quindi dalla (131) si deduce, facendo divergere r,

$$0 \equiv \int_\sigma \{A(M)((\partial F(M,Q)/\partial \nu) - L(M)F(M,Q)) - B(M)F(M,Q)\}\,d\sigma - \int_\tau f(M)F(M,Q)\,d\tau$$

cioè le funzioni A, B soddisfano alla (15).

Sia ora R interno a τ; cioè si ponga $R = P$. Per ipotesi, la funzione $u(M)$ è sommabile in τ; per il Lemma III, lo è anche la funzione $u(M)MP^{-k}$, con $k < m$, per tutti i punti P di τ in cui è verificata la (128), e cioè quasi ovunque in τ.

Sia P uno di tali punti. Risulta, per la (2),

$$(132)\quad E^*(F_r(M,P)) = \Sigma a_{ik}(M)\,\frac{\partial^2 F_r(M,P)}{\partial x_i \partial x_k} + \Sigma b^*_i(M)\,\frac{\partial F_r(M,P)}{\partial x_i} + c^*(M)F_r(M,P)$$

$$= \Sigma a_{ik}(P)\,\frac{\partial^2 F_r(M,P)}{\partial x_i \partial x_k} + \Sigma(a_{ik}(M) - a_{ik}(P))\,\frac{\partial^2 F_r(M,P)}{\partial x_i \partial x_k} + \Sigma b_i^*(M)\,\frac{\partial F_r(M,P)}{\partial x_i} + c^*(M)F_r(M,P)$$

e osserviamo che, essendo

[13] L. Tonelli, "Sulla rappresentazione analitica della funzioni di più variabili reali," *Rendiconti del Circolo Matematico di Palermo*, vol. 29 (1910), p. 14 e pp. 20-24.

$$F(M,P) = O(MP^{-(m-2)}),$$

$$\frac{\partial F(M,P)}{\partial x_i} = O(MP^{-(m-1)}),$$

$$\frac{\partial^2 F(M,P)}{\partial x_i \partial x_k} = O(MP^{-m}),$$

uniformemente al variare di M, P in τ, per il lemma II le stesse limitazioni valgono per F_r, $\partial F_r/\partial x_i$, $\partial^2 F_r/\partial x_i \partial x_k$, uniformemente rispetto a r ed al variare di M, P in τ.

Dalla (132), indicato con E_0 l'operatore differenziale (a coefficienti indipendenti da $x_1, \cdots, x_m$) $\Sigma a_{ik}(P)(\partial^2/\partial x_i \partial x_k)$, segue perciò

$$E^*(F_r(M,P)) = E_0(F_r(M,P)) + \chi_r(M,P) \tag{133}$$

e si ha

$$|\chi_r(M,P)| \leq KMP^{-(m-1)} \tag{134}$$

con K costante positiva indipendente da r, M, P.

Siccome $u(M)MP^{-(m-1)}$ è sommabile in τ, si ricava allora dalla (134), per il teorema di Lebesgue sull'integrazione per serie,

$$\lim_{r\to\infty} \int_\tau u(M)\chi_r(M,P)\,d\tau = \int_\tau u(M)\{\Sigma(a_{ik}(M) - a_{ik}(P))\frac{\partial^2 F(M,P)}{\partial x_i \partial x_k} + \Sigma b^*_i(M)\,\frac{\partial F(M,P)}{\partial x_i} + c^*(M)F(M,P)\}\,d\tau = -\int_\tau u(M)E_0(F(M,P))\,d\tau, \tag{135}$$

essendo $F(M,P)$ un integrale della (4).

Si ha poi, per $M \neq P$ e indicato con E'_0 l'operatore $\Sigma a_{ik}(P)(\partial^2/\partial y_i \partial y_k)$,

$$E_0(F_r(M,P)) = \Sigma a_{ik}(P)(\partial^2/\partial x_i \partial x_k)\int_{\tau'} F(N,P)\phi_r(N,M)\,d\tau' = \int_{\tau'} F(N,P)E'_0(\phi_r(N,M))\,d\tau'. \tag{136}$$

Sia τ'_R un dominio ipersferico col centro in P e raggio $\epsilon < MP$, σ'_ϵ l'ipersfera (P,ϵ), ν'_0 la direzione conormale, relativa all'operatore E'_0, orientata verso l'interno di $\tau' - \tau'_P$.

Si ha, per la formula di Green,

$$(137)\quad -\int_{\tau'-\tau'_{\iota'}}\{F(N,P)E'_0(\phi_r(N,M))-\phi_r(N,M)E'_0(F(N,P))\}d\tau'$$
$$=\int_{\sigma'}\{F(N,P)\frac{\partial\phi_r(N,M)}{\partial\nu'_0}-\frac{\partial F(N,P)}{\partial\nu'_0}\phi_r(N,M)\}d\sigma'$$
$$+\int_{\sigma'_{\iota'}}\{F(N,P)\frac{\partial\phi_r(N,M)}{\partial\nu'_0}-\frac{\partial F(N,P)}{\partial\nu'_0}\phi_r(N,M)\}d\sigma'_P$$

e risulta

$$(138)\quad \lim_{\epsilon\to 0}\int_{\sigma'_P}\{F(N,P)\frac{\partial\phi_r(N,M)}{\partial\nu'_0}-\frac{\partial F(N,P)}{\partial\nu'_0}\phi_r(N,M)\}d\sigma'_P$$
$$=k_m\phi_r(P,M).$$

Siccome è $|E'_0(F(N,P))|\leqq K'NP^{-(m-1)}$, con K' costante positiva, segue allora dalle (136), (137), (138)

$$(139)\quad E_0(F_r(M,P))=-k_m\phi_r(P,M)$$
$$+\int_{\sigma'}\{\phi_r(N,M)\frac{\partial F(N,P)}{\partial\nu'_0}-\frac{\partial\phi_r(N,M)}{\partial\nu'_0}F(N,P)\}d\sigma'$$
$$+\int_{\tau'}\phi_r(N,M)E'_0(F(N,P))d\tau'.$$

Ora, in virtù della (128) ed essendo $\phi(P,M)=\phi(M,P)$, risulta

$$(140)\quad \lim_{r\to\infty}\int_\tau u(M)\phi_r(P,M)d\tau=u(P).$$

Si ha poi, essendo τ interno a τ',

$$(141)\quad \lim_{r\to\infty}\int_\tau u(M)[\int_{\sigma'}\{\phi_r(N,M)\frac{\partial F(N,P)}{\partial\nu'_0}-\frac{\partial\phi_r(N,M)}{\partial\nu'_0}F(N,P)\}d\sigma']d\tau=0$$

e infine, per l'osservazione relativa al Lemma II, essendo $E'_0(F(M,P))=O(MP^{-(m-1)})$,

$$(142)\quad \lim_{r\to\infty}\int_\tau u(M)[\int_{\sigma'}\phi_r(N,M)E'_0(F(N,P))d\tau']d\tau$$
$$=\int_\tau u(M)E_0(F(M,P))d\tau.$$

Dalle (133), (134), (135), (139), (140), (141), (142), segue allora

$$\lim_{r\to\infty}\int_\tau u(M)E^*(F_r(M,P)d\tau=-k_m u(P)$$

e quindi, per la (131), facendo divergere r e tenendo presente il lemma II,

$$k_m u(P) = \int_\sigma \{A(M)((\partial F(M,P)/\partial \nu) - L(M)F(M,P)) - B(M)F(M,P)\}d\sigma - \int_\tau f(M)F(M,P)d\tau$$

cioè le funzioni u, A, B soddisfano alla (16).

COROLLARIO. *La successione* $\{E^*(w_r)\}$ *è chiusa rispetto alla totalità delle funzioni sommabili in* τ.

Infatti se risulta, per ogni r,

$$\int_\tau uE^*(w_r)d\tau = 0$$

si soddisfa al sistema (130) ponendo $A \equiv B \equiv f \equiv 0$. Ne segue, per la (16), $u(P) = 0$ quasi ovunque in τ.

Osservazione. In luogo delle funzioni w_r date dalle (129) si possono porre nella (130) le funzioni di una qualsiasi successione $\{w'_r\}$ tali che w_r e le sue derivate parziali prime e seconde si possano approssimare uniformemente in τ, contorno σ incluso, rispettivamente mediante combinazioni lineari della w' e mediante le derivate parziali prime e seconde di queste combinazioni.

Di tale proprietà gode una larga classe di successioni $\{w'_r\}$ chiuse rispetto alla totalità delle funzioni sommabili in un dominio τ' contente τ nel suo interno. Ad esempio, si può porre

$$w'_r = e^{2\pi i \Sigma \alpha_{k,r}(x_k/T_k)}, \qquad (\alpha_{k,r} = 0, 1, \cdots)$$

essendo $T_k > \delta_k$, massimo valore della differenza $|x_k - x'_k|$ al variare dei punti $M(x_1, \cdots, x_m)$, $M'(x'_1, \cdots, x'_m)$ in τ.

4. Con le stesse notazioni adoperate nel **4** del I, considerando il problema misto, si ottiene dal sistema (130) il sistema

$$(143)\quad \int_\tau uE^*(w_r)d\tau + \int_{\sigma_1} \{A((\partial w_r/\partial \nu) - w_r(L - (h/k))\}d\sigma_1 - \int_{\sigma_2} B\left\{\frac{k}{h}\frac{\partial w_r}{\partial \nu} + w_r(1 - L(k/h))\right\}d\sigma_2 = \int_{\sigma_1}(C/k)w_r d\sigma_1 - \int_{\sigma_2}(C/h((\partial w_r/\partial \nu) - Lw_r)d\sigma_2 + \int_\tau f w_r d\tau = c_r$$

dove le costanti c_r sono note.

Definito allora un vettore $\boldsymbol{G}$ di componenti u in τ, A in σ_1, B in σ_2 e una successione di vettori $\{\boldsymbol{\omega}_r\}$ di componenti $E^*(w_r)$ in τ, $\{(\partial w_r/\partial \nu) - w_r(L - (h/k))\}$ in σ_1, $-\{\frac{k}{h}\frac{\partial w_r}{\partial \nu} + w_r(1 - L(k/h))\}$ in σ_2, si ricava, per la (143),

$$(\boldsymbol{G}, \boldsymbol{\omega}_r) = c_r \tag{144}$$

cioè sono noti i coefficienti di Fourier dell'incognito vettore $\boldsymbol{G}$ rispetto alla successione $\{\boldsymbol{\omega}_r\}$. Viceversa, sia $\boldsymbol{G}$ una soluzione del sistema (144). Si conoscono allora le funzioni u in τ, A su σ_1, B su σ_2 *e*, per le (110), (111) le B, A possono definirsi su tutto σ; si trova poi, eliminando C dalle (143) mediante le (110), (111), che u, A, B soddisfano al sistema (130) e quindi per il teorema III alle (15), (16). Per il teorema I e per le (110), (111) risulta perciò soddisfatta su σ la (109).

Inoltre, se vale, nell'insieme Γ_a, il teorema di unicità per il problema considerato, la successione $\{\boldsymbol{\omega}_r\}$ è *chiusa* rispetto alla totalità dei vettori $\boldsymbol{G}$ di componenti sommabili; se tale teorema non vale, le autosoluzioni sono *caratterizzate* dall'essere i corrispondenti vettori $\boldsymbol{G}$ ortogonali alla successione $\{\boldsymbol{\omega}_r\}$. Infine, se il problema non ammette soluzione, nemmeno il sistema (144) ammette soluzione.

Dedotta poi dalla successione $\{\boldsymbol{\omega}_r\}$ una equivalente successione $\{\boldsymbol{\xi}_r\}$ ortogonale e normale, il sistema (144) equivale al sistema

$$(\boldsymbol{G}, \boldsymbol{\xi}_r) = d_r$$

dove le costanti d_r risultano note. Perciò *condizione necessaria e sufficiente perchè il problema ammetta soluzione, il vettore* $\boldsymbol{G}$ *risultando di norma integrale finita, è che converga la serie* Σd^2_r.

Considerazioni del tutto analoghe si possono svolgere per il problema di Cauchy, nel quale il vettore incognito $\boldsymbol{G}$ ha componenti u in τ, A in σ_2, B in σ_2 e il vettore $\boldsymbol{\omega}_r$ ha componenti $E^*(w_r)$ in τ, $(\partial v_r/\partial \nu) - Lv_r$ in σ_2, $-v_r$ in σ_2.

VIA POLESINE, 20,
ROME, ITALY.

SUR UNE SUITE DE QUADRIQUES ASSOCIÉE A UNE CONGRUENCE *W*.*

par LUCIEN GODEAUX.

Dans une note récente, M. Chenkuo Pa [1] a cherché à donner une nouvelle définition de la suite de quadriques que nous avons attachée à tout point non parabolique d'une surface; [2] il a ensuite indiqué que l'on pouvait de même attacher une suite de quadriques à toute droite d'une congruence W. C'est ce que nous avons fait voici déjà quelques années [3] et nous voudrions indiquer brièvement les résultats auxquels nous sommes parvenus.

1. Soit (x) une surface rapportée à ses asymptotiques u, v. Désignons par Q l'hyperquadrique de Klein, appartenant à un espace linéaire S_5, à cinq dimensions, qui représente les droites de l'espace, par U, V les points de Q qui représentent respectivement les tangentes en un point x de (x) à la ligne u (sur laquelle u varie) et à la ligne v passant par ce point. Tzitzeica [4] et M. Bompiani [5] ont montré que les points U, V sont les transformés de Laplace l'un de l'autre. Ils appartiennent donc à une suite de Laplace

$$(1) \qquad \cdots, U_n, \cdots, U_1, U, V, V_1, \cdots, V_n, \cdots,$$

chaque point étant le transformé du précédent dans le sens des u. Cette suite este autopolaire par rapport à Q; le point U_n, par exemple, est le pôle de l'hyperplan $V_{n-2}V_{n-1}V_nV_{n+1}V_{n+2}$. Il en résulte que les plans $U_nU_{n+1}U_{n+2}$ et $V_nV_{n+1}V_{n+2}$ sont conjugués par rapport à Q; ils coupent cette hyperquadrique suivant deux coniques qui représentent les deux séries réglées d'une quadrique Φ_n. Pour $n = 0, 1, 2, \cdots$, on obtient la suite de quadriques que nous avons

* Received April 21, 1947.

[1] "A new definition of the Godeaux sequence of quadrics," *American Journal of Mathematics*, vol. 69 (1947), pp. 117-120.

[2] "Sur les lignes asymptotiques d'une surface et l'espace réglé," *Bulletin de l'Académie royale de Belgique* (1927), pp. 812-826; (1928), pp. 31-41; "La théorie des surfaces et l'espace réglé," *Actualités scientifiques et industrielles*, Paris (1934).

[3] "Sur quelques familles de quadriques associées aux points d'une surface," *Annales de la Société Polonaise de Mathématique* (1928), pp. 213-226; "La théorie des surfaces . . .," *loc. cit.*, pp. 21-24.

[4] *Géométrie projective différentielle des réseaux*, Paris, (1924).

[5] "Sull'equazione di Laplace," *Rendiconti del Circolo Matematico di Palermo*, vol. 34 (1912), pp. 383-407.

attachée au point x de la surface, la première quadrique étant la quadrique de Lie. Deux quadriques consécutives de la suite se touchent en quatre points qui sont caractéristiques pour les deux quadriques.

La section de Q par l'hyperplan $V_{n-2}V_{n-1}V_nV_{n+1}V_{n+2}$ représente un complexe linéaire et les coordonnées du pôle U_n de l'hyperplan par rapport à Q sont les coefficients de l'équation de ce complexe en coordonnées de droites. On obtient donc une suite de complexes linéaires qui se succèdent dans une suite de Laplace. On peut facilement, au moyen de cette suite de complexes, définir la suite de quadriques Φ, Φ_1, Φ_2, $\cdots$, sans passer par l'espace S_5.

2. Soient j une droite engendrant une congruence $W:(j)$ et (x), $(\bar{x})$ les surfaces focales de cette congruence. Les asymptotiques u, v se correspondent sur les surfaces (x), $(\bar{x})$ et nous attacherons à la surface (x) la suite le Laplace (1), à la surface $(\bar{x})$ la suite de Laplace analogue

$$(2) \qquad \cdots, \bar{U}_n, \cdots, \bar{U}_1, \bar{U}, \bar{V}, \bar{V}_1, \cdots, \bar{V}_n, \cdots.$$

Les droites UV, $\bar{U}\bar{V}$ se coupent en un point J, représentant sur Q la droite j de (j) et on sait (Darboux) que le point J décrit un réseau conjugué aux congruences de droites UV, $\bar{U}\bar{V}$. Le point J appartient donc à une suite de Laplace

$$(3) \qquad \cdots, J_n, \cdots, J_1, J, J_{-1} \cdots, J_{-n}, \cdots,$$

inscrite dans les suites (1) et (2). D'une manière précise, le point J_n, par exemple, est le point d'intersection des droites $U_{n-1}U_n$ et $\bar{U}_{n-1}\bar{U}_n$.

Les droites $U\bar{U}$, $V\bar{V}$ se coupent en un point P qui est le pôle par rapport à Q de l'hyperplan $J_2J_1JJ_{-1}J_{-2}$. Le point P appartient à une suite de Laplace

$$(4) \qquad \cdots, P_n, \cdots, P_1, P, P_{-1}, \cdots, P_{-n}, \cdots,$$

polaire de la suite (3) par rapport à Q. La suite (4) est circonscrite aux suites (1) et (2). Dans les suites (3) et (4), chaque point est le transformé du précédent dans le sens des u et le point P_n est le pôle de l'hyperplan $J_{-n+2}, J_{-n+1}, J_{-n}, J_{-n-1}, J_{-n-2}$.

Les plans $J_{-n-1}J_{-n}J_{-n+1}$ et $P_{n-1}P_nP_{n+1}$ sont conjugués par rapport à Q et coupent cette hyperquadrique suivant deux coniques qui représentent les génératrices des deux modes d'une quadrique Ψ_n. Lorsque n prend toutes les valeurs entières, positives, nulle ou négatives, on obtient une suite de qua-

driques associées à la génératrice j de la congruence (j). Pour $n = 0$, la quadrique Ψ_0 dégénère en deux plans qui sont les plans focaux de droite j.

Deux quadriques consécutives de la suite

$$(5) \qquad \cdots, \Psi_{-n}, \cdots, \Psi_{-1}, \Psi_0, \Psi_1, \cdots, \Psi_n, \cdots$$

se touchent en quatre points qui sont des points caractéristiques de ces quadriques.

En remarquant que les coordonnées du point P_n sont les coefficients de l'équation d'un complexe linéaire, on obtient une suite de complexes linéaires se succédant dans une suite de Laplace; cette suite comprend le complexe linéaire osculateur à la congruence (j) le long de la droite j. On pourra en déduire la définition de la suite de quadriques (5) sans passer par l'espace S_5.

Nous renvoyons à nos notes citées plus haut pour d'autres propriétés.[6]

UNIVERSITÉ DE LIÉGE.

[6] Voir aussi: Rozet, "Recherches sur les congruences W," *Mémoires de la Société des Sciences de Liége* (1935), pp. 1-31.

SKEW SETS.*

By R. H. Bing.

A skew set is one which is not topologically equivalent to any subset of the surface of a sphere. Kuratowski proved [1] that a compact skew continuous curve which contains only a finite number of simple closed curves contains one of the two following types of curves:

Type 1. A skew curve S_1 is of type 1 provided there exist six distinct points P_1, P_2, P_3, Q_1, Q_2, Q_3 such that S_1 is the sum of nine arcs $P_1Q_1, P_1Q_2, \cdots, P_3Q_3$ with end points as indicated and with the common part of two of these arcs that intersect each other being an end point of each.

Type 2. A skew curve S_2 is of type 2 provided there exist five distinct points P_1, P_2, P_3, P_4, P_5 such that S_2 is the sum of ten arcs $P_1P_2, P_1P_3 \cdots, P_4P_5$ with end points as indicated and with the common part of two of these arcs that intersect each other being an end point of each.

Claytor showed [2] that a compact cyclic continuous curve is skew only if it contains one of these two types of curves. It would be interesting to get a simple characterization of a general planar set. The study of skew sets may lead in that direction.

In this paper we shall consider the following two types of skew sets which are generalizations of the types of skew curves studied by Kuratowski:

Skew set of type 1. The set S_1 is a skew set of type 1 provided there exist six distinct points P_1, P_2, P_3, Q_1, Q_2, Q_3 such that S_1 is the sum of nine connected sets $M(P_1Q_1), M(P_1Q_2), \cdots, M(P_3Q_3)$ where $M(P_iQ_j)$ contains $P_i + Q_j$ and contains a limit point of $M(P_rQ_s)$ only if $P_i + Q_j$ intersects $P_r + Q_s$.

Skew set of type 2. The set S_2 is a skew set of type 2 provided there exist five points P_1, P_2, P_3, P_4, P_5 such that S_2 is the sum of ten connected sets $M(P_1P_2), M(P_1P_3), \cdots, M(P_4P_5)$ where $M(P_iP_j)$ contains $P_i + P_j$ and contains a limit point of $M(P_rP_s)$ only if $P_i + P_j$ intersects $P_r + P_s$.

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The justification for calling these skew sets follows from Theorems 3 and 4.

Because of their lengths, the proofs of the following two theorems are omitted.

THEOREM 1. *Suppose that P_1, P_2, P_3, Q_1, Q_2, Q_3 are six distinct points and that $P_1Q_1, P_1Q_2, \cdots, P_3Q_3$ are nine arcs with end points as indicated and such that two of these arcs intersect each other only if they have an end point in common. Then the sum of these arcs is not homeomorphic to any plane set.*

THEOREM 2. *Suppose that P_1, P_2, P_3, P_4, P_5 are five distinct points such that $P_1P_2, P_1P_3, \cdots, P_4P_5$ are ten arcs with end points as indicated and such that two of these arcs intersect each other only if they have an end point in common. Then the sum of these arcs is not homeomorphic to any plane set.*

The space S considered in the next two theorems has the property that any pair of points of a component of an open set in S can be joined by an arc lying in that open set.

THEOREM 3. *If S contains a skew set of type* 1, *it contains a skew curve of type* 1.

Proof. Let $M(P_1Q_1), M(P_1Q_2), \cdots, M(P_3Q_3)$ be sets in S satisfying the conditions mentioned in the definition of a skew set of type 1. Let $D(P_1Q_1)$ be an open set containing $M(P_1Q_1)$ but no point of $M(P_2Q_2) + M(P_2Q_3) + M(P_3Q_3)$ and let P_1Q_1 be an arc from P_1 to Q_1 in $D(P_1Q_1)$. If we replace $M(P_1Q_1)$ by the arc P_1Q_1 in the sequence $M(P_1Q_1), M(P_1Q_2), \cdots, M(P_3Q_3)$, the sum of the resulting sequence is a skew set of type 1. Similarly, we may replace $M(P_1Q_2)$ by an arc P_1Q_2, $M(P_1Q_3)$ by an arc $P_1Q_3, \cdots$, and $M(P_3Q_3)$ by an arc P_3Q_3 in such a way that two of the arcs $P_1Q_1, P_1Q_2, \cdots, P_3Q_3$ intersect each other only if they have an end point in common.

We shall now show that $P_1Q_1 + P_1Q_2 + \cdots + P_3Q_3$ contains nine arcs $P_1O_{11}Q_1, P_1O_{12}Q_2, \cdots, P_3O_{33}Q_3$ with end points as indicated such that two of these arcs intersect each other only if they have an end point in common and the common part of two of these arcs that intersect each other has only a finite number of components.

Let P_1Q_1 be $P_1O_{11}Q_1$. We shall determine $P_1O_{12}Q_2$ as follows. Since $P_1O_{11}Q_1 \cdot P_1Q_2$ is a closed and compact set, it can be covered by a finite set of components of $P_1O_{11}Q_1 - P_1O_{11}Q_1 \cdot (P_2Q_1 + P_3Q_1)$. Let $P_1O_{12}Q_2$ be an arc from P_1 to Q_2 in P_1Q_2 plus the sum of these components such that $P_1O_{11}Q_1 \cdot P_1O_{12}Q_2$ has only a finite number of components.

Now $P_1O_{11}Q_1 \cdot P_1Q_3$ can be covered by a finite set T_1 of components of $P_1O_{11}Q_1 - P_1O_{11}Q_1 \cdot (P_2Q_1 + P_3Q_1)$ and $P_1O_{12}Q_2 \cdot P_1Q_3$ can be covered by a finite set T_2 of components of $P_1O_{12}Q_2 - P_1O_{12}Q_2 \cdot (P_2Q_2 + P_3Q_2)$. Let $P_1O_{13}Q_3$ be an arc from P_1 to Q_3 in P_1Q_3 plus the sum of the elements of $T_1 + T_2$ such that each of the sets $P_1O_{11}Q_1 \cdot P_1O_{13}Q_3$ and $P_1O_{12}Q_2 \cdot P_1O_{13}Q_3$ has only a finite number of components.

This process is continued. To get $P_3O_{33}Q_3$, we let T_{ij} be a finite collection of components of $P_iO_{ij}Q_j - P_iO_{ij}Q_j \cdot (P_1O_{11}Q_1 + P_1O_{12}Q_2 + P_2O_{21}Q_1 + P_2O_{22}Q_2)$ covering $P_iO_{ij}Q_j \cdot P_3Q_3$ and let $P_3O_{33}Q_3$ be an arc from P_3 to Q_3 in P_3Q_3 plus the sum of the elements of $T_{13} + T_{23} + T_{31} + T_{32}$ such that the common part of $P_3O_{33}Q_3$ and each of the sets $P_1O_{13}Q_3$, $P_2O_{23}Q_3$, $P_3O_{31}Q_1$, $P_3O_{32}Q_2$ has only a finite number of components. Denote the collection of arcs $P_1O_{11}Q_1, P_1O_{12}Q_2, \cdots, P_3O_{33}Q_3$ by G and the sum of the elements of G by G^*. We note that two elements of G intersect each other only if they have an end point in common and the common part of two intersecting elements of G has only a finite number of components.

Some subset of G^* is irreducible with respect to containing six distinct points P'_1, P'_2, P'_3, Q'_1, Q'_2, Q'_3 and nine arcs $P'_1Q'_1, P'_1Q'_2, \cdots, P'_3Q'_3$ such that two of these arcs intersect each other only if they have an end point in common. For convenience, suppose that G^* is such a subset of itself.

No point of G^* is a point of order five or more. If there were such a point P', it would belong to some three elements of G and the point P' could be used instead of the common end point of the three arcs of G containing P'. Then G^* would not be irreducible in the sense supposed. Likewise, we find that G^* contains no point of order four that belongs to three elements of G.

Since the common part of no two elements of G has infinitely many components, G^* does not contain infinitely many simple closed curves. We know by Theorem 1 that G^* is a skew set and by the previously mentioned theorem of Kuratowski that it contains either a skew curve of type 1 or a skew curve of type 2.

Assume that G^* contains a skew curve of type 2. Let $A_1, A_2, \cdots, A_5$ be five distinct points and $A_1A_2, A_1A_3, \cdots, A_4A_5$ be ten arcs in G^* such that the common part of two of these arcs that intersect each other is an end point of each. If some component of $G^* - (A_1 + A_2 + \cdots + A_5)$ has three of

The justification for calling these skew sets follows from Theorems 3 and 4.

Because of their lengths, the proofs of the following two theorems are omitted.

THEOREM 1. *Suppose that P_1, P_2, P_3, Q_1, Q_2, Q_3 are six distinct points and that $P_1Q_1, P_1Q_2, \cdots, P_3Q_3$ are nine arcs with end points as indicated and such that two of these arcs intersect each other only if they have an end point in common. Then the sum of these arcs is not homeomorphic to any plane set.*

THEOREM 2. *Suppose that P_1, P_2, P_3, P_4, P_5 are five distinct points such that $P_1P_2, P_1P_3, \cdots, P_4P_5$ are ten arcs with end points as indicated and such that two of these arcs intersect each other only if they have an end point in common. Then the sum of these arcs is not homeomorphic to any plane set.*

The space S considered in the next two theorems has the property that any pair of points of a component of an open set in S can be joined by an arc lying in that open set.

THEOREM 3. *If S contains a skew set of type 1, it contains a skew curve of type 1.*

Proof. Let $M(P_1Q_1), M(P_1Q_2), \cdots, M(P_3Q_3)$ be sets in S satisfying the conditions mentioned in the definition of a skew set of type 1. Let $D(P_1Q_1)$ be an open set containing $M(P_1Q_1)$ but no point of $M(P_2Q_2) + M(P_2Q_3) + M(P_3Q_3)$ and let P_1Q_1 be an arc from P_1 to Q_1 in $D(P_1Q_1)$. If we replace $M(P_1Q_1)$ by the arc P_1Q_1 in the sequence $M(P_1Q_1), M(P_1Q_2), \cdots, M(P_3Q_3)$, the sum of the resulting sequence is a skew set of type 1. Similarly, we may replace $M(P_1Q_2)$ by an arc P_1Q_2, $M(P_1Q_3)$ by an arc $P_1Q_3, \cdots$, and $M(P_3Q_3)$ by an arc P_3Q_3 in such a way that two of the arcs $P_1Q_1, P_1Q_2, \cdots, P_3Q_3$ intersect each other only if they have an end point in common.

We shall now show that $P_1Q_1 + P_1Q_2 + \cdots + P_3Q_3$ contains nine arcs $P_1O_{11}Q_1, P_1O_{12}Q_2, \cdots, P_3O_{33}Q_3$ with end points as indicated such that two of these arcs intersect each other only if they have an end point in common and the common part of two of these arcs that intersect each other has only a finite number of components.

Let P_1Q_1 be $P_1O_{11}Q_1$. We shall determine $P_1O_{12}Q_2$ as follows. Since $P_1O_{11}Q_1 \cdot P_1Q_2$ is a closed and compact set, it can be covered by a finite set of components of $P_1O_{11}Q_1 - P_1O_{11}Q_1 \cdot (P_2Q_1 + P_3Q_1)$. Let $P_1O_{12}Q_2$ be an arc from P_1 to Q_2 in P_1Q_2 plus the sum of these components such that $P_1O_{11}Q_1 \cdot P_1O_{12}Q_2$ has only a finite number of components.

Now $P_1O_{11}Q_1 \cdot P_1Q_3$ can be covered by a finite set T_1 of components of $P_1O_{11}Q_1 - P_1O_{11}Q_1 \cdot (P_2Q_1 + P_3Q_1)$ and $P_1O_{12}Q_2 \cdot P_1Q_3$ can be covered by a finite set T_2 of components of $P_1O_{12}Q_2 - P_1O_{12}Q_2 \cdot (P_2Q_2 + P_3Q_2)$. Let $P_1O_{13}Q_3$ be an arc from P_1 to Q_3 in P_1Q_3 plus the sum of the elements of $T_1 + T_2$ such that each of the sets $P_1O_{11}Q_1 \cdot P_1O_{13}Q_3$ and $P_1O_{12}Q_2 \cdot P_1O_{13}Q_3$ has only a finite number of components.

This process is continued. To get $P_3O_{33}Q_3$, we let T_{ij} be a finite collection of components of $P_iO_{ij}Q_j - P_iO_{ij}Q_j \cdot (P_1O_{11}Q_1 + P_1O_{12}Q_2 + P_2O_{21}Q_1 + P_2O_{22}Q_2)$ covering $P_iO_{ij}Q_j \cdot P_3Q_3$ and let $P_3O_{33}Q_3$ be an arc from P_3 to Q_3 in P_3Q_3 plus the sum of the elements of $T_{13} + T_{23} + T_{31} + T_{32}$ such that the common part of $P_3O_{33}Q_3$ and each of the sets $P_1O_{13}Q_3$, $P_2O_{23}Q_3$, $P_3O_{31}Q_1$, $P_3O_{32}Q_2$ has only a finite number of components. Denote the collection of arcs $P_1O_{11}Q_1, P_1O_{12}Q_2, \cdots, P_3O_{33}Q_3$ by G and the sum of the elements of G by G^*. We note that two elements of G intersect each other only if they have an end point in common and the common part of two intersecting elements of G has only a finite number of components.

Some subset of G^* is irreducible with respect to containing six distinct points P'_1, P'_2, P'_3, Q'_1, Q'_2, Q'_3 and nine arcs $P'_1Q'_1, P'_1Q'_2, \cdots, P'_3Q'_3$ such that two of these arcs intersect each other only if they have an end point in common. For convenience, suppose that G^* is such a subset of itself.

No point of G^* is a point of order five or more. If there were such a point P', it would belong to some three elements of G and the point P' could be used instead of the common end point of the three arcs of G containing P'. Then G^* would not be irreducible in the sense supposed. Likewise, we find that G^* contains no point of order four that belongs to three elements of G.

Since the common part of no two elements of G has infinitely many components, G^* does not contain infinitely many simple closed curves. We know by Theorem 1 that G^* is a skew set and by the previously mentioned theorem of Kuratowski that it contains either a skew curve of type 1 or a skew curve of type 2.

Assume that G^* contains a skew curve of type 2. Let $A_1, A_2, \cdots, A_5$ be five distinct points and $A_1A_2, A_1A_3, \cdots, A_4A_5$ be ten arcs in G^* such that the common part of two of these arcs that intersect each other is an end point of each. If some component of $G^* - (A_1 + A_2 + \cdots + A_5)$ has three of

the five points $A_1, A_2, \cdots, A_5$ as limit points, a careful examination of G^* shows that it contains a skew set of type 1. Compare this notion with a theorem by Hall [3]. Assume that no such component contains three of these points as limit points.

Let X be a component of $G - (A_1 + A_2 + \cdots + A_5)$ that contains P_1 and has A_i and A_j as limit points. If BA_i and CA_i are arcs in $X + A_i$ each of which is a subset of an element of G, then BA_i and CA_i do not belong to different elements of G or else A_i is a point of order four or more belonging to three elements of G.

There are arcs from P_1 to Q_1 in each of the sets $P_1O_{11}Q_1$, $P_1O_{12}Q_2 + P_2O_{22}Q_2 + P_2O_{21}Q_1$, and $P_1O_{13}Q_3 + P_3O_{33}Q_3 + P_3O_{31}Q_1$ but no two of these sets contains an arc lying in $X + A_i$ and containing A_i; also no two of these sets contains an arc lying in $X + A_j$ and containing A_j. Therefore Q_1 is a point of X. Likewise, we find that each of the six points P_1, P_2, P_3, Q_1, Q_2, Q_3 is a point of X. But $G^* - X$ is not a subset of one arc. Hence, G^* contains a skew curve of type 1.

The following theorem may be established in a like manner.

THEOREM 4. *If S contains a skew set of type 2, it contains either a skew curve of type 1 or a skew curve of type 2.*

Example. That we cannot conclude under the hypotheses of Theorem 4 that S contains a skew curve of type 2 may be seen from the following example showing that a skew curve of type 1 contains a skew set of type 2. Consider the skew curve which is the sum of nine arcs $P_1Q_1, P_1Q_2, \cdots, P_3Q_3$ satisfying conditions given in the definition of a skew curve of type 1. Let P_1Q_1 be the sum of the two arcs P_1R and RQ_1. The sum of the arcs $RQ_1 + P_2Q_1$, $RQ_1 + P_3Q_1$, $P_1R + P_1Q_2$, $P_1R + P_1Q_3$, $P_2Q_1 + P_3Q_1$, P_2Q_2, P_2Q_3, P_3Q_2, P_3Q_3, and $P_1Q_2 + P_1Q_3$ is a skew set of type 2. However, it contains no skew curve of type 2.

THEOREM 5. *Suppose that $\alpha_1, \alpha_2, \cdots, \alpha_n$ are arcs in the plane such that two of these arcs intersect only if they have an end point in common. There exists a collection of arcs $\beta_1, \beta_2, \cdots, \beta_n$ in the plane such that β_i $(i = 1, 2, \cdots, n)$ has the same end points α_i and contains no non end point of β_j $(j \neq i)$.*

Proof. Let γ_i $(i = 1, 2, \cdots, n)$ be an arc (not necessarily in the plane) having the same end points as α_i and such that γ_i contains no non end point

of γ_j $(j \neq i)$. We shall show that $\gamma_1 + \gamma_2 + \cdots + \gamma_n$ is topologically equivalent to a plane set. The truth of Theorem 5 will follow as a consequence.

If $\gamma_1 + \gamma_2 + \cdots + \gamma_n$ is not homeomorphic with a plane set, it contains a skew curve of either type 1 or type 2. Assume that it contains nine arcs $P_1Q_1, P_1Q_2, \cdots, P_3Q_3$ satisfying conditions given in the definition of a skew curve of type 1. Then P_iQ_j is the sum of a subcollection of the arcs $\gamma_1, \gamma_2, \cdots, \gamma_n$; suppose that it is the sum of $\gamma_r, \cdots$, and γ_s. Let $M(P_iQ_j)$ be the sum of the arcs $\alpha_r, \cdots$, and α_s. Then the sum of $M(P_1Q_1), M(P_1Q_2), \cdots$, and $M(P_3Q_3)$ is a skew set of type 1. But the plane contains no skew set of type 1. Also, the assumption that $\gamma_1 + \gamma_2 + \cdots + \gamma_n$ contains a skew curve of type 2 leads to the contradiction that $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ contains a skew set of type 2.

Question. It would be interesting to know if the following statement is true: If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are arcs two of which intersect only if they have an end point in common, then there exist arcs $\beta_1, \beta_2, \cdots, \beta_n$ such that two of these arcs intersect only if they have an end point in common, the common part of two of these arcs that intersect is connected, and β_i $(i = 1, 2, \cdots, n)$ is an arc in $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ having the same end points as α_i.

THEOREM 6. *No plane set G contains a collection of five mutually separated points sets such that the closure of the sum of any pair of these five sets is the closure of a connected subset of G which is open in G.*

Proof. Assume that the theorem is false. Let S_1, S_2, S_3, S_4, S_5 be five mutually separated sets in the plane set G such that the closure of the sum of any pair of these five sets is the closure of a connected subset of G which is open in G. Denote a point of S_i $(i = 1, 2, 3, 4, 5)$ by P_i. Let T_{ij} $(i = 1, 2, 3, 4, 5;\ j = 1, 2, 3, 4, 5)$ be a connected subset of G which is open in G and such that the closure of T_{ij} is the closure of $S_i + S_j$. If $M(P_iP_j)$ is the sum of P_i, P_j, and T_{ij}, the sum of the sequence $M(P_1P_2), M(P_1P_3), \cdots, M(P_4P_5)$ is a skew set of type 2. But the plane contains no skew set of type 2 because it contains no skew curve of either type 1 or type 2.

Problem. The following problem could be posed to a person not acquainted with terms in topology. Consider a nation composed of more than four states. Suppose that there are five political parties, that each state favors some one of these parties, and that each party is favored by some state. Would it be possible for each pair of the parties to hold a joint

convention at some place so that each of the states favoring these two parties could send a delegate to the convention so that this delegate could remain in the states but not cross the boundary of any "unfriendly" state, that is one favoring a party other than one of the two holding the joint convention.

We see by Theorem 6 that the answer is in the negative. We let the sum of the states be G and the sum of the interiors of the states favoring a particular party be one of the five sets.

THE UNIVERSITY OF TEXAS.

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FREE LOOPS AND NETS AND THEIR GENERALIZATIONS.*

By GRACE E. BATES.**

INTRODUCTION.

It is the object of this investigation to develop a chapter of loop theory [1] that is parallel to the theory of free Abelian groups and direct sums of Abelian groups, and to the theory of free groups and free products of groups. Like these theories, our study is concerned with two problems: the existence of free loops and of free sums of loops (with and without amalgamated subloops), and the structure of their subloops. In this we succeed fairly completely in so far as we are able to prove the existence of the desired loops in a very comprehensive fashion (even more so than in group theory), and in so far as we are able to prove the theorems analogous to the subgroup theorems of Schreier, Kurosch, and others.

In Section **1**, we develop an independent study of free nets (and their generalizations), and this theory, in a somewhat restricted form, is applied in later sections to obtain the basic loop theorems. The greater generality of the net theory seems to indicate the possibility of a wider application of this theory than we have used in this paper. In the study of isotopy, for example, it has been shown [2] that loops are isotopic (similar) if and only if their associated nets are isomorphic, which suggests that the net approach would be preferable in this study.

In Section **2** we generalize the theorems of Baer and Bol [3] to show the equivalence of our theory of half-nets and half-loops. In particular, we prove that this equivalence applies also to homomorphisms in the two theories.

We are able to state in **3**, then, as almost immediate consequences of preceding net theorems, the basic theorems on free loops and their generaliza-

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[1] For references on loop theory, see Albert (1), (2); Baer (4); Bruck (1), (2); Smiley (1).

The numbers in parentheses refer to the Bibliography at the end of this paper.

[2] See Baer (3); the term "isotopy" is due to Albert, see Albert (1), (2).

[3] See Baer (3) and Bol (1).

tions. In particular, we have the theorem that any half-loop is embeddable in one and essentially only one loop which is freely generated by it. The fundamental theorem of this section is the Subloop Theorem, which has as a corollary the loop analogue to Schreier's theorem. We are able to generalize this theorem to free sums of loops and to obtain a refinement theorem on the decomposition of a loop into free summands.

Throughout Section **3** references are made to corresponding theorems from group theory—a procedure which is later justified in the Appendix. Several interesting contrasts between the two theories are noted. For example, every free loop (except the loop of order one) contains a free subloop of countably infinite rank, whereas a like statement about groups is obviously false for free cyclic groups. Again, although not every semi-group can be embedded in a group, it is true that every semi-loop can be embedded in a loop. Indeed, for any semi-loop which is not itself a loop, the containing loop (generated by the semi-loop) is not at all uniquely determined as is the case for Abelian semi-groups. It is also noteworthy that in many of the loop theorems which have their analogues in group theory, the statements for loops are sharper than those possible for groups—as, for example the theorem on subloops of a free sum of loops and the Generalized Subloop Theorem.

The Appendix contains a general formulation of the concept of "freeness" which comprehends the classical definitions of free group and free Abelian group, etc., as well as the definitions used in this paper for loops.

Notations. If J and K are subsets of the loop L, we denote by $J \cup K$, the set-theoretical sum of J and K; by $J + K$, the subloop generated by J and K; and by $J \cap K$, the cross-cut of J and K.

If x is an element of the loop L, and if σ is a homomorphism of L, the image of x under σ is denoted by $x\sigma$; similarly, we denote by $J\sigma$ the image of the subset J of L. We use the notation $L \sim K$, if L is isomorphic to K.

We apply similar notations to nets.

1. NETS AND HALF-NETS.

1.1. Basic concepts and definitions. A *half-net* consists of four different kinds of elements: points, r-lines, s-lines, and t-lines, with the following incidence relations:

(i) Through every point there passes at most one line of each pencil. (By a *pencil* is meant the set of all lines of any one of the three types, as, for example, the set of all r-lines.)

(ii) Two lines meet in at most one point.

It is a consequence of (i) that lines in the same pencil do not meet.

In general, we shall use capitals for points and lower case letters for lines; we write $P < h$ for "P lies on h."

A *net* is a half-net which satisfies the additional requirements:

(i′) Through every point there passes at least one line of each pencil.

(ii′) Two lines of different pencils meet in at least one point.

It is clear that any subset of a net, under the same incidence relations as those prevailing in the net, constitutes a *sub-half-net* of the net.

The above definitions of half-nets and nets permit a set of lines, all of the same pencil, to be considered a half-net (or a net). We exclude, however, these "degenerate nets" from all our considerations.

Definition. If H is a sub-half-net of the half-net K, then H is *closed* in K if the following conditions are satisfied:

(a) If the point P is in H, and if P lies on h in K, then h is in H.

(b) If the lines $h \neq h'$ are in H, and if h, h', carry Q in K, then Q is in H.

Clearly, any subnet of a net N is closed in N.

A half-net K is *generated* by its sub-half-net H, if K is the only sub-half-net of K which contains H and is closed in K.

A *homomorphism* ϕ of a half-net M is a single-valued mapping of the elements of M upon the elements of a half-net K, such that points are mapped on points, x-lines are mapped on x-lines, and P on h implies that $P\phi$ is on $h\phi$. It is easily verified that the homomorphic image of a net is a net.

A homomorphism ϕ of a half-net M *upon* the half-net K is an *isomorphism* if ϕ is $1-1$ and if ϕ^{-1} is a homomorphism of K upon M. One-to-oneness is enough to insure that a homomorphism of a net upon another net is an isomorphism, but that this is not the case for proper half-nets is illustrated by the following example: Let M be the half-net consisting of the r-line, h, and the point P not on h, and let K be the half-net consisting of the r-line, k, and the point Q on k. Then M may be mapped $1-1$ and homomorphically upon K, but the inverse map is clearly not a homomorphism of K upon M.

THEOREM 1.1. *If ρ and σ are homomorphisms of the half-net M, (into*

a half-net K), if M is generated by its sub-half-net J, and if $j\rho = j\sigma$, for every j in J, then $\rho = \sigma$.

Proof. Let T be the sub-half-net of M consisting of those elements, x, of M for which $x\rho = x\sigma$. Clearly, T contains J. Furthermore, T is closed in M, for: Suppose that P is in T, and $P < h$ in M; then in K, $P\rho < h\sigma$ and $P\sigma < h\sigma$. But $P\rho = P\sigma$, and in K there is at most one line of the same pencil as h on the point $P\rho = P\sigma$. Hence $h\rho = h\sigma$, and h is in T. Similarly, if $h \neq h'$ are in T, and if h, h', carry Q in M, then we have $h\rho = h\sigma$ and $h'\rho = h'\sigma$ on $Q\rho$ and on $Q\sigma$ in K, thus implying that $Q\rho = Q\sigma$, and hence Q is in T. Since M is generated by J, and $J \subset T$, then $M = T$; that is, $\rho = \sigma$.

1. 2. Extension chains. A half-net M is an *L-extension* of its sub-half-net J, if:

(1) Every point of M is in J.

(2) Any line of M which is not in J lies on at least one point of J.

If M satisfies the additional requirement that every point of M carries at least one line of each pencil, then M is a *complete L-extension* of J.

A half-net K is a *P-extension* of its sub-half-net J, if:

(1) Every line of K is in J.

(2) Any point of K which is not in J lies on at least two lines of J.

If K satisfies the additional requirement that every pair of lines of different pencils in K meet in at least one point, then K is a *complete P-extension* of J.

The chain $J = J_0 \subset J_1 \subset \cdots \subset J_{2n} \subset J_{2n+1} \subset J_{2n+2} \subset \cdots$, where J_{2n+1} is an L-extension of J_{2n}, and J_{2n+2} is a P-extension of J_{2n+1}, will be called an *extension chain* of J. If the J_i are complete extensions, we call the chain a *maximal extension chain* of J. We use this notation throughout.

It is almost obvious that any sub-half-net J of a net N has a maximal extension chain within N. For, if J_{2n} is constructed in N, then J_{2n+1} may be constructed in N by virtue of the fact that every point of $J_{2n} \subset N$ has one and only one line of each pencil in N; and if J_{2n+1} is constructed in N, then J_{2n+2} may be constructed in N, since every pair of lines of different pencils in $J_{2n+1} \subset N$ meet in one and only one point in N. Hence, by induction, N contains $J = J_0 \subset J_1 \subset \cdots \subset J_i \subset \cdots$. Furthermore, it is clear that this maximal extension chain of J within N is uniquely determined by N.

Note that our definition of an extension chain does not require that the J_i be contained in a net, although it will be shown later that, as a consequence of the definition, this is the case.

We denote by $\bigcup J_i = M$, the set-theoretical join of the J_i, where the J_i are terms in an extension chain of $J = J_0$, and we define incidence in M as follows: If P, h, are in M, then $P < h$ in M if there exists an i such that P, h, are in J_i with $P < h$ in J_i. (Note that $P < h$ in J_i implies that $P < h$ in J_{i+k}.) It is clear that, under these incidence relations, $M = \bigcup J_i$ is a half-net. Furthermore, if the J_i form a *maximal* extension chain, then M satisfies the additional requirements for a net, since any element x in M is in some J_i and consequently in all succeeding J_i. Thus, P in M is in some J_{2k+1} in which every point has at least one line of each pencil, and h, h', of different pencils in M, are both in some J_{2p+2} in which every pair of lines of different pencils meet in at least one point.

Definition. When M, a half-net, can be written in the form: $M = \bigcup J_i$, for some $J = J_0$ contained in M, and with J_{i+1} an L or P-extension of J_i for every i, then we say that J and M are connected by the chain of J_i's.

LEMMA 1.1. *K is a half-net generated by the half-net J, if and only if, J and K are connected by an extension chain of J.*

Proof. If J and K are connected by an extension chain of J, then $K = \bigcup J_i$, and K is a half-net. We have only to show that K is generated by $J = J_0$. Suppose that there exists a sub-half-net T of K such that $J \subset T \subset K$, and that T is closed in K. Assume that it has already been proved that $J_i \subset T$. If $J_{i+1} = J_i$, then $J_{i+1} = J_i \subset T$; hence suppose that $J_{i+1} \neq J_i$. We distinguish two cases:

(1) $i = 2n$. Let h be a line in J_{2n+1} which is not in J_{2n}. Then h is on at least one point P of J_{2n}. But P is in T, and T is closed in K; hence h is in T.

(2) $i = 2n + 1$. Let P be a point in J_{2n+2} which is not in J_{2n+1}. Then P is on at least two lines k, k', of J_{2n+1}. Since k, k', are in T, and T is closed in K, then P is in T.

By induction, we have $\bigcup J_i = K \subset T \subset K$, and hence $T = K$. That is, K is generated by J.

Conversely, suppose that K is a half-net generated by J; then we may construct in K an extension chain whose first term is J, as follows:

Let J_{2i+1} contain J_{2i} together with all those lines of K which are on

points of J_{2i}, and let J_{2i+2} contain J_{2i+1} together with all those points of K which are intersections of two or more lines of J_{2i+1}. That is, we might call J_{i+1} a K-complete extension of J_i. Then it is easily verified that the union of the J_i is closed in K, so that we have $K = \bigcup J_i$.

COROLLARY. *N is a net generated by the half-net J, if and only if, N and J may be connected by a maximal extension chain.*

Definition. A *null net* is a net consisting of one point and the three lines, one of each pencil, through this point. A *null half-net* is a half-net consisting of one of the following combinations of elements: (1) one point and no lines; (2) one line and one point on this line; (3) two lines of different pencils and no points; (4) two lines of different pencils and one point which is on these two lines; (5) a null net.

It is easily verified that in any net not a null net, there are at least two lines of each pencil, and every line carries at least two points.

Definition. M, an L-extenion of J, is *open* if it satisfies the additional requirement:

(2′) Any line of M which is not in J lies on *at most* one point of J.

This requirement, together with condition (2) for an L-extension, implies that any line of M which is not in J, lies on one and only one point of J.

K, a P-extension of J is open if it satisfies the additional requirement:

(2′) Any point of K which is not in J lies on *at most* two lines of J.

This requirement, together with condition (2) for a P-extension, implies that any point of K which is not in J, lies on two and only two lines of J.

The chain, $J = J^{(0)} \subset J^{(1)} \subset \cdots \subset J^{(2n)} \subset J^{(2n+1)} \subset J^{(2n+2)} \subset \cdots$, where $J^{(2n+1)}$ is an open L-extension of $J^{(2n)}$, and $J^{(2n+2)}$ is an open P-extension of $J^{(2n+1)}$, will be called an *open extension chain* of J. If the $J^{(i)}$ are *complete* extensions, we call the above chain a *maximal open extension chain.* We shall follow this notation throughout.

Given a half-net J, we may construct a maximal open extension chain of which J is the first member, in the following way[4]: If P in J lacks an x-line, we adjoin to J one and only one x-line, $x(P)$, in such a way that $x(P)$ carries P and no other point of J. This we do for every point P in J, and for $x = r, s, t$. It is obvious from the nature of this construction that $J^{(1)}$,

[4] See Hall (1), p. 236. Our construction is similar to Hall's construction for completing a partial projective plane. However, it will be apparent later that our construction has the advantage of uniqueness in regard to net homomorphisms.

the set of elements in J, together with the lines just adjoined, is a half-net under the incidence relations of J together with those incurred by the method of adding new lines. Furthermore, since no new points are adjoined in $J^{(1)}$, but every point of J has in $J^{(1)}$ one line of each pencil, it is clear that $J^{(1)}$ is a complete L-extension. We then form $J^{(2)}$ from $J^{(1)}$ by adjoining to each pair of lines h, h', not meeting in $J^{(1)}$, one and only one point Q, requiring Q to lie on h and h', but on no other line of $J^{(1)}$. Again it is clear that $J^{(2)}$ is a half-net, and that $J^{(2)}$ constitutes a complete P-extension of $J^{(1)}$. This process may be carried out a denumerable number of times, with each step producing a bona-fide half-net. Thus $J^{(2k+1)}$ is formed by adjoining to $J^{(2k)}$ lines h on points P of $J^{(2k)}$ in the manner indicated above, so that in $J^{(2k+1)}$ every point has one (and only one) line of each pencil; and $J^{(2k+2)}$ is formed from $J^{(2k+1)}$ by adjoining points of intersection Q to lines h, h', of $J^{(2k+1)}$, in the manner indicated above, so that in $J^{(2k+2)}$ every pair of lines of different pencils meet in one and only one point.

Then by the Corollary to Lemma 1.1, if we form $\bigcup J^{(i)}$, we are assured of a net $N = \bigcup J^{(i)}$, which is generated by J. That is, *any half-net may be embedded, in at least one way, in a net.*

For future reference, we state below several properties of maximal open extension chains. We exclude the null-net and null-half-nets from these considerations.

Let $J^{(0)} \subset J^{(1)} \subset \cdots \subset J^{(i)} \subset \cdots$ be a maximal open extension chain, and let $N = \bigcup J^{(i)}$; then N is a net generated by $J^{(0)}$, from the above discussion. We enumerate the following properties for this chain:

(1°) If $P \neq Q$ are in $J^{(i)}$, for some i, and if P and Q lie on the same line h in N, then P and Q lie on h in $J^{(i)}$; and if h, h', h'', all of different pencils, are in some $J^{(j)}$, and h, h', h'', lie on a point P in N, then they lie on P in $J^{(j)}$.

This property of the chain is obvious from the nature of our construction, and is also valid for partial open extension chains.

(2°) $J^{(i)} = J^{(i+1)}$, for some $i > 0$, if and only if, $J^{(0)} = N$.

If $J^{(0)} = N$, it is obvious that $J^{(i)} = J^{(0)}$ for all i, so that we need only to justify the statement in one direction. Suppose that $J^{(i)} = J^{(i+1)}$, for some $i > 0$. Then we have $J^{(2k)} = J^{(2k+1)}$ or $J^{(2k)} = J^{(2k-1)}$, for some $k > 0$. But in $J^{(2k\pm1)}$ every point carries one and only one line of each pencil, and in $J^{(2k)}$, $k > 0$, every pair of lines of different pencils meet in one and only one point. Hence $J^{(i)} = J^{(i+1)}$ is a net, closed in N, and since

$J^{(0)} \subset J^{(i)} \subset N$, with N generated by $J^{(0)}$, we must have $J^{(i)} = N$. Without loss in generality, suppose that $i = 2k$. Then $J^{(2k-1)} = J^{(2k)} = N$; for, if there were a point in $J^{(2k)}$, but not in $J^{(2k-1)}$, then this point would lie on two and only two lines in $J^{(2k)} = N$, which is impossible, since in a net every point carries three lines. But then $J^{(2k-2)} = J^{(2k-1)} = N$; for, if there were a line x in $J^{(2k-1)}$, but not in $J^{(2k-2)}$, then x would lie on one and only one point in $J^{(2k-1)} = N$, which is impossible, since in a non-null net every line carries at least two points. Since k is a positive integer, we must eventually reach the stage in which $J^{(0)} = J^{(1)} = N$, proving our statement.

As an immediate consequence of property (2°), note that if $J^{(0)} \neq N$, then N has an infinite number of lines and points.

(3°) For i sufficiently large, and $J^{(0)} \neq N$, in $J^{(2i+1)}$ and $J^{(2i+3)}$ lines of each of the three pencils are added.

Since every non-null net has at least two lines of each pencil, then, for some $i > 0$, $J^{(2i)}$ has at least two lines of each pencil. Since $J^{(2i+1)} \neq J^{(2i)}$, there is at least one line adjoined in $J^{(2i+1)}$. Suppose, for definiteness, that this is an r-line, r'; then in $J^{(2i+1)}$ there must be at least one s-line, say s', and at least one t-line, say t', which do not meet r'. In $J^{(2i+2)}$, then, the points $P' = r's'$, and $Q' = r't'$ are adjoined, so that in $J^{(2i+3)}$ the t-line, $t(P')$, and the s-line, $s(Q')$ are adjoined.

(4°) If $J^{(0)} \neq N$, and if p is any positive integer, then there is an integer j, such that the number of points adjoined in $J^{(2j)}$ is greater than p, and the number of lines adjoined in $J^{(2j+1)}$ is greater than p.

For i sufficiently great, $J^{(2i-1)}$ contains at least two lines of each pencil. Suppose that in $J^{(2i)}$, there are k points adjoined; (by (2°), k is necessarily greater than 0). Then in $J^{(2i+1)}$, k lines are adjoined, one on each of the k new points of $J^{(2i)}$. In $J^{(2i+1)}$, for each of these k lines there must be at least two lines, one of each of the remaining two pencils, which do not meet the line just adjoined. Hence in $J^{(2i+2)}$, there must be adjoined at least $2k$ points, and in $J^{(2i+3)}$, at least $2k$ lines. That is, for i sufficiently large, the number of points adjoined in the P-extensions is unbounded, as is the number of lines adjoined in the L-extensions.

(5°) For i sufficiently large, and $J^{(0)} \neq N$, there are points P, Q, with $P \neq Q$, such that P and Q are in $J^{(2i)}$, but not in $J^{(2i-1)}$, satisfying in $J^{(2i)}$ the following conditions: (1) Both P and Q lack an x-line (x, being any given one of the three pencils); (2) P and Q have no common line.

By (4°), for j sufficiently great, there are four different points, R, S, T, U,

which are in $J^{(2j)}$, but not in $J^{(2j-1)}$. Each of these points is on two and only two lines in $J^{(2j)}$, and hence to each of these points there corresponds one and only one pencil x $(= r, s, t)$, such that this point is not on any x-line. Since there exist but three pencils, then at least two points among these four, say R, S, are on lines of the same pencils. We assume, without loss in generality, that neither R nor S is on an r-line. If R and S have no common line, then we have found two points satisfying the requirements of our statement. If R and S lie on the same s-line, then in $J^{(2j+2)}$, the points $P = r(R) \cdot t(S)$ and $Q = t(R) \cdot r(S)$ are adjoined, and clearly, both P and Q lack an s-line, and have no line in common. Similarly, if R and S have, instead, a common t-line, then $P' = r(R) \cdot s(S)$ and $Q' = s(R) \cdot r(S)$ in $J^{(2j+2)}$ satisfy the requirements of our statement.

1.3. Existence of free extensions. We make the following definition:

Definition. A half-net K is *free over its sub-half-net* J, if every homomorphism of J into a net N may be extended to a homomorphism of K into N. If a half-net K is free over and generated by its sub-half-net J, we shall say that K is *freely generated* by J.

LEMMA 1.2. *Given a half-net J, then there is essentially at most one net which is freely generated by J.*

Proof. Suppose that N, M, are nets which are free over and generated by J. Then we may extend the identity homomorphism of $J \subset N$ upon $J \subset M$ to a homomorphism ϕ of N upon $N\phi \subset M$. Since M is generated by J, then $N\phi = M$. But we may also extend the identity map of $J \subset M$ upon $J \subset N$ to a homomorphism ψ of M upon $M\psi \subset N$, and again we have $M\psi = N$. Then $N\phi\psi = N$, and $j\phi\psi = (j\phi)\psi = j\psi = j$, for all j in J. By Theorem 1.1, then $\phi\psi = 1$. Similarly, we have $\psi\phi = 1$. That is, ϕ and ψ are reciprocal isomorphisms of N and M, and we have $N \sim M$.

LEMMA 1.3. *If the half-net R is freely generated by its sub-half-net S, and if the sub-half-net T of R contains S and is generated by S, then*: (1) *R is freely generated by T, and* (2) *T is free over S.*

Proof. (1) Given a homomorphism ψ of T, then ψ induces a homomorphism ψ_S of S. Since R is free over S, ψ_S may be extended to a homomorphism ϕ of R, and ϕ induces ϕ_T in T. But we have, then, $s\phi_T = s\phi = s\psi_S = s\psi$, for every s in S. Hence (Theorem 1.1), $\phi_T = \psi$ and ϕ is an extension of ψ to a homomorphism of R; that is, R is free over T. Clearly, R is generated by T, since R is generated by $S \subset T$.

(2) Let σ be a homomorphism of S; then σ may be extended to a homomorphism ρ of R, and ρ induces ρ_T in T. We have $s\sigma = s\rho_T$, for every s in S, and hence ρ_T is an extension of σ to a homomorphism of T. That is, T is free over S.

The next two lemmas are preparatory to the proof of the existence of a net which is freely generated by a half-net.

LEMMA 1.4. (a). *K, an L-extension of the half-net J, is free over J, if and only if, K is an open L-extension.*

Proof. We assume, first, that K is an open L-extension.

Given η, a homomorphism of J into a net N, we define a mapping ϕ of K into N as follows: Let ϕ act on elements of J in the same manner as η. Consider an element x which is in K, but not in J; then x is a line in K, lying on one and only one point P of J, since K is an open extension. In N, $P\phi = P\eta$ has one and only one line h of the same pencil as x; we define $x\phi = h$. That ϕ, as so defined, is single-valued is a consequence of the fact that x lies on P and on no other point in K. Clearly, then, ϕ is a homomorphism of K into N, and ϕ is an extension of η. Hence K is free over J.

Now assume that K is free over J.

Suppose that K is not an open extension of J. Then there must be at least one line h in K, but not in J, such that h lies on P, Q, of J, with $P \neq Q$. But now consider the net $N = \bigcup J^{(i)}$, where $J^{(0)} = J$, and where $J^{(i+1)}$ is a complete open extension (L or P), of $J^{(i)}$. Take the identity map of $J \subset K$ upon $J \subset N$. Since K is free over J, this map may be extended to a homomorphism ψ of K into N. In N, $h\psi$ carries $P\psi = P$, and $Q\psi = Q$. But, since P and Q are both in J, $P \neq Q$, then $h\psi$ is in J on P and Q (Property (1°) for maximal open extension chains). Hence P (and Q) in J had already one line $h\psi$ of the same pencil as h. Then $h = h\psi$, and h is in J, which is a contradiction to our hypothesis. Hence K is open.

LEMMA 1.4. (b). *M, a P-extension of J, is free over J, if and only if, M is an open P-extension of J.*

Proof. Assume that M is an open P-extension of J.

Given η, a homomorphism of J into a net N, we define a mapping ϕ of M into N as follows: Let ϕ act on elements of J in the same manner as η. Consider an element x which is in M, but not in J; then x is a point of M, lying on two and only two lines, h, h', of different pencils in J. The lines $h\phi = h\eta$ and $h'\phi = h'\eta$ in N have one and only one point of intersection, Q.

Define $x\phi = Q$. That ϕ as so defined is single-valued, follows from the fact that x lies on h, h', and on no other line in M. Then, clearly, ϕ is a homomorphism of M into N, and ϕ coincides with η in J; hence M is free over J.

Now assume that M is free over J.

Suppose that M is not an open P-extension. Then there must be at least one point P in M, but not in J, which lies on three different lines, h, h', h'', of J. Consider the net $N = \bigcup J^{(i)}$, where $J^{(0)} = J$, and where $J^{(i+1)}$ is a complete open (L or P) extension of $J^{(i)}$. The identity map of $J \subset M$ upon $J \subset N$ may be extended to a homomorphism ψ of M into N, since M is free over J. In N, $P\psi$ lies on $h\psi, h'\psi, h''\psi$,—that is, on h, h', h''; since h, h', h'', are all in J, then $P\psi$ is in J (Property (1°) for maximal open extension chains). Hence h, h', h'', intersect already in one point $P\psi$ of $J \subset M$. Then $P\psi = P$, and P is in J, which is a contradiction to our hypothesis. Hence M is open.

LEMMA 1.5. *N is a half-net which is freely generated by the half-net J, if and only if, N may be represented in the form: $N = \bigcup J^{(i)}$, where $J^{(0)} = J$, and where $J^{(i+1)}$ is an open (L or P) extension of $J^{(i)}$.*

Proof. Assume, first, that we have $N = \bigcup J^{(i)}$; by previous arguments, then N is a half-net which is generated by $J = J^{(0)}$. Hence, we have only to show that N is free over J.

Given η, a homomorphism of J into a net M, suppose that η has already been extended to a homomorphism η_{2n} of $J^{(2n)}$ into M. By Lemma 1.4 (a), then η_{2n} may be extended to a homomorphism η_{2n+1} of $J^{(2n+1)}$ into M, and by Lemma 1.4. (b), η_{2n+1} may be extended to a homomorphism η_{2n+2} of $J^{(2n+2)}$ into M. Hence, by induction, there exists, for every i, a homomorphism η_i of $J^{(i)}$, such that η_i and η_{i+k} coincide in $J^{(i)}$. We define a mapping ϕ of N into M as follows: If x is in N, then x is in $J^{(i)}$ for some i, and we define $x\phi = x\eta_i$. Then ϕ is single-valued, since for x in $J^{(i)}$ and in $J^{(j)}$, $i < j$, we have $x\eta_i = x\eta_j = x\phi$. Furthermore, ϕ is a homomorphism of N, since $P < h$ in N implies that $P < h$ in some $J^{(i)}$, and hence $P\eta_i = P\phi$ lies on $h\eta_i = h\phi$. Hence, since η has been extended to a homomorphism of N, N is free over J.

Conversely, suppose that there is given a half-net N which is freely generated by its sub-half-net J. By Lemma 1.1, then N may be represented in the form $N = \bigcup J_i$, with $J = J_0$ and where J_{i+1} is an L or P-extension of J_i. We wish to show that these are open extensions.

For any J_i in the chain, we have $J \subset J_i \subset N$, and by hypothesis N is free over J. Since J_i is generated by J, J_i is free over J (Lemma 1.3,(2)).

But then, for every i, we have $J \subset J_i \subset J_{i+1}$, where J_{i+1} is free over J_i (Lemma 1.3,(1)). By Lemmas 1.4,(a) and (b), then, the J_i are open extensions; that is, in our notation, we have $J_i = J^{(i)}$ and $N = \bigcup J^{(i)}$.

COROLLARY. *N is a net which is freely generated by the half-net J, if and only if, N may be represented in the form: $N = \bigcup J^{(i)}$, where $J^{(0)} = J$, and $J^{(i+1)}$ is a complete open (L or P) extension of $J^{(i)}$.*

This corollary is an immediate consequence of Lemma 1.5 and the corollary to Lemma 1.1.

THEOREM 1.2. *Any half-net J may be embedded in one and essentially only one net N which is freely generated by J.*

We have already noted that any half-net J may be embedded in a net N, where N may be represented in the form: $N = \bigcup J^{(i)}$ (see the construction following the Corollary to Lemma 1.1). Then Theorem 1.2 is an immediate consequence of the corollary to Lemma 1.5 and Lemma 1.2.

THEOREM 1.3. *Given a half-net J which is not a net and not a null-half-net, then there exists an extension of J to a net M which is generated by J, but not free over J* (Thus, there exist non-equivalent extensions of (proper) half-nets to nets).

Proof. Form the maximal open extension chain of J:

$$J = J^{(0)} \subset J^{(1)} \subset \cdots \subset J^{(i)} \quad \cdots,$$

thus obtaining the net $N = \bigcup J^{(i)}$, which is freely generated by J. Since $J \neq N$, by property (5°) for maximal open extension chains there exist, for i sufficiently large, points $P \neq Q$, such that P and Q are in $J^{(2i)}$, but not in $J^{(2i-1)}$, where P, Q, satisfy conditions: (1) Both P and Q lack an x-line, (2) P and Q have no common line. For convenience, suppose that P and Q lack an r-line.

Then we may form the half-net J^*, which is obtained from $J^{(i)}$ by adjoining one r-line connecting P and Q. Clearly, J^* is not free over $J^{(2i)}$ (Lemma 1.4,(b)), and, if we extend J^* to the net M which is freely generated by J^*, then M is generated by J, but is not free over J (Corollary to Lemma 1.5).

1.4. Free nets and free sums of nets. If the net N is freely generated by its sub-half-net J, and if J contains lines, but no points, then we define J as a *free set of generators* of N. A *free net*, then, is a net having a free set of generators.

From Theorem 1.2, it is clear that, given any set of lines J (not all of the same pencil), there is one and essentially only one net N for which J is a free set of generators. We note also, since there are no incidence relations in a free set of generators J of a net N, that any single-valued mapping (which, of course, maps x-lines on x-lines), of J into a net may be extended to a homomorphism of N into that same net.

THEOREM 1.4. *Any net is the homomorphic image of a free net.*

Proof. Consider any net M; let x be any line in M, belonging, say, to the pencil of s-lines. Then x, together with the set of all points of M which lie on x, form a sub-half-net K of M which generates M. For, every point of M not on x is connected by lines from different pencils with points on x.

Let J be the set consisting of the line x, and all the r-lines through points of x (i.e. J is the set of all r-lines in M, together with the s-line x). Extend J to the free net S having J as a free set of generators. If $J \subset S$ is mapped by the identity map upon $J \subset M$, this map may be extended to a homomorphism ϕ of S into M, and since J generates M, then $M = S\phi$.

Definition. A net N is the *free sum of its subnets* $N(i)$, finite or infinite in number, if the following conditions are satisfied:

(1) N is generated by $\bigcup N(i)$.

(2) If, for every i, there is a homomorphism $\phi(i)$ of $N(i)$ into the net M, then there exists a homomorphism ϕ of N into M, which coincides with $\phi(i)$ in $N(i)$.

We use the notation $N = \overset{*}{\sum} N(i)$ to represent the free sum of the $N(i)$. We write $N(1) * N(2)$ to designate the free sum of $N(1)$ and $N(2)$, and similarly for any finite number of free summands.

An immediate consequence of this definition is that $N(i) \cap N(j)$ is vacuous for all $i \neq j$, where the $N(i)$ are free summands. For, if $N(i)$ and $N(j)$ have a common point, then we map every point P in $N(i)$ upon one and the same point $P\phi(i)$ in M, and all the x-lines in $N(i)$ upon the x-lines through $P\phi(i)$ in M; and at the same time, we map every point Q in $N(j)$, for all $j \neq i$, upon one and the same point $Q\phi(j) \neq P\phi(i)$, and all the x-lines in these $N(j)$ upon the x-lines on $Q\phi(j)$ in M. It is obvious that $\phi(i)$ and the $\phi(j)$ cannot be extended simultaneously to a homomorphism of $N = \overset{*}{\sum} N(i)$.

Remark. If $N = \overset{*}{\sum} N(i)$, then $N = N(i) * M$, where M is the net generated by $\bigcup_{j \neq i} N(j)$, (Theorem 1.1).

If a net M is the free sum of nets $M(i)$, where each $M(i)$ is a free net, then M is a free net. For, each $M(i)$ has a free set of generators, $J(i)$, and it is easily verified that $J = \bigcup J(i)$ is a free set of generators of M.

1.5. Structure of subnets of free nets. We are able to determine, (essentially), completely the structure of subnets of free nets (and their generalizations) by means of the following theorem:

THEOREM 1.5. *If the net N is freely generated by its sub-half-net J, and if S is a subnet of N, then S is the free sum of the nets F and H, where:*

i. *H is freely generated by $S \cdot J$.*

ii. *F is a free net with a free set of generators:*

$$F_0 = \bigcup_{i=0}^{\infty} \{(S \cdot J^{(2i+1)}) - (S \cdot J^{(2i)})_L\}, \text{ where}$$

$J = J^{(0)} \subset J^{(1)} \subset \cdots \subset J^{(2i)} \subset J^{(2i+1)} \subset J^{(2i+2)} \subset \cdots$ *is the maximal (open) extension chain of J in N.*

The proof requires several lemmas. Throughout these discussions we use the subscript, "L," to denote a complete L-extension within N of a sub-half-net of N, and the subscript, "P," to denote a complete P-extension.

LEMMA (a). *If $J_0 \subset J_1 \subset \cdots \subset J_i \subset \cdots$ is a maximal extension chain of J_0, and if S is a subnet of the net $N = \bigcup_{j=0}^{\infty} J_i$, then:*

(1) $(S \cdot J_{2n})_L \subset S \cdot J_{2n+1}$

(2) $(S \cdot J_{2n+1})_P = S \cdot J_{2n+2}$ (for all n).

Proof. (1) Clearly $S \cdot J_{2n} \subset S \cdot J_{2n+1}$. Consider an element x, which is in $(S \cdot J_{2n})_L$, but not in $S \cdot J_{2n}$; then x is a line in J_{2n+1} on at least one point P of $S \cdot J_{2n}$. Since P is in S, a subnet of N, then x is also in S; hence x is in $S \cdot J_{2n+1}$, proving (1).

(2) Clearly $S \cdot J_{2n+1} \subset S \cdot J_{2n+2}$. Consider an element y in $(S \cdot J_{2n+1})_P$, but not in $S \cdot J_{2n+1}$; then y is a point in J_{2n+2} on at least two lines h, h', of $S \cdot J_{2n+1}$. Since h, h', are both in the subnet S, then y is in S. Hence y is in $S \cdot J_{2n+2}$, and we have $(S \cdot J_{2n+1})_P \subset S \cdot J_{2n+2}$. If x is an element in

$S \cap J_{2n+2}$, then, if x is also in J_{2n+1}, x is in $S \cap J_{2n+1} \subset (S \cap J_{2n+1})_P$; hence, suppose that x is not in J_{2n+1}. Then x is a point in J_{2n+2} on at least two lines h, h', of J_{2n+1}. But, since x is in the subnet S, h and h' are in S, so that we have h, h', in $S \cap J_{2n+1}$. Then x is in $(S \cap J_{2n+1})_P$ and thus $S \cap J_{2n+2} \subset (S \cap J_{2n+1})_P$. Hence, $(S \cap J_{2n+1})_P = S \cap J_{2n+2}$.

COROLLARY. *Under the hypotheses of the Lemma, if J_{2n+1} is an open L-extension of J_{2n}, then $(S \cap J_{2n})_L$ is an open L-extension of $S \cap J_{2n}$; and if J_{2n+2} is an open P-extension of J_{2n+1}, then $S \cap J_{2n+2}$ is an open P-extension of $S \cap J_{2n+1}$.*

This is an immediate consequence of the fact that an element in $(S \cap J_{2n})_L$, but not in $S \cap J_{2n}$, is in J_{2n+1} but not in J_{2n}; and an element in $S \cap J_{2n+2}$ but not in $S \cap J_{2n+1}$ is in J_{2n+2} but not in J_{2n+1}.

If K and M are sub-half-nets of a net N, such that $K \subset M$, then we denote by $M - K$, the half-net consisting of elements which are in M but not in K, with incidence relations for these elements as prescribed by N.

We say that two sub-half-nets J, K, of a net N are *totally disconnected* if $J \cap K$ is vacuous, and if we have:

(i) If P is in J, h in K, then $P \not< h$ (in N).

(ii) If k is in J, Q in K, then $k \not> Q$ (in N).

Hence, if we have a set of half-nets $J(n)$, which are pairwise totally disconnected in N, then $\bigcup J(n)$ is a sub-half-net of N, which has the property that $P < h$ in $\bigcup J(n)$, if and only if, $P < h$ in $J(n)$, for some n.

LEMMA (b). *If $J_0 \subset J_1 \subset \cdots \subset J_i \subset \cdots$ is a maximal extension chain generating the net $N = \bigcup J_i$, and if S is a subnet of N, then*

$$K(i) = \{(S \cap J_{2i+1}) - (S \cap J_{2i})_L\}, \quad i = 0, 1, 2, \cdots$$

has the following properties:

(1) *$K(i)$ is a set of lines from S, and contains no points.*

(2) *$K(i) \cap K(j)$ is vacuous for $i \neq j$.*

(3) *$(S \cap J_{2n})_L$ and $\bigcup_{i \geq n} K(i)$ are totally disconnected.*

Proof. (1) By Lemma (a), $(S \cap J_{2i})_L \subset S \cap J_{2i+1}$, so that the expressions for $K(i)$ have meaning. Clearly, $K(i) \subset S$. If x is in $K(i)$ then x is in $S \cap J_{2i+1}$, but not in $(S \cap J_{2i})_L$. But then x is in J_{2i+1} and not in J_{2i}, so that x is a line.

(2) Since $K(i) \subset J_{2i+1} - J_{2i}$, it is obvious that $K(i) \cdot K(j)$ must be vacuous for $i \neq j$.

(3) The set $(S \cdot J_{2n})_L$ is excluded from $K(n)$; since we have $(S \cdot J_{2n})_L \subset S \cdot J_{2n+1}$, it is also excluded from all the $K(i)$, for $i > n$. Hence $(S \cdot J_{2n})_L \cdot \bigcup_{i \geq n} K(i)$ is vacuous. Now consider a point P in $(S \cdot J_{2n})_L$; since $(S \cdot J_{2n})_L$ is a complete L-extension of $S \cdot J_{2n}$ then $r(P)$, $s(P)$, $t(P)$, are all in $(S \cdot J_{2n})_L$, and hence not in $\bigcup_{i \geq n} K(i)$. That is, for P in $(S \cdot J_{2n})_L$, there is no h in $\bigcup_{i \geq n} K(i)$, such that $P < h$. Since $\bigcup_{i \geq n} K(i)$ contains no points, there is no Q in $\bigcup_{i \geq n} K(i)$, such that $Q < k$, for k in $(S \cdot J_{2n})_L$. Hence we have proved (3).

LEMMA (c). *If $J_0 \subset J_1 \subset \cdots \subset J_i \subset \cdots$ is a maximal extension chain generating the net $N = \bigcup J_i$, and if S is a subnet of N, then S is generated by*

$$M = (S \cdot J_0) \cdot \bigcup_{i \geq 0} \{(S \cdot J_{2i+1}) - (S \cdot J_{2i})_L\}.$$

Proof. Define:

$$M(2n) = (S \cdot J_{2n}) \cdot \bigcup_{i \geq n} K(i); \qquad M(2n+1) = (S \cdot J_{2n+1}) \cdot \bigcup_{i \geq n+1} K(i),$$

where, as in Lemma (b), $K(i) = \{(S \cdot J_{2i+1}) - (S \cdot J_{2n})_L\}$. Thus $M = M(0)$.

S may be represented in the form: $S = \bigcup (S \cdot J_i)$—for, clearly $\bigcup (S \cdot J_i) \subset S$, and S is also contained in $\bigcup (S \cdot J_i)$, since x in $S \subset N = \bigcup J_i$ is, for some n, in $S \cdot J_n \subset \bigcup (S \cdot J_i)$.

Now consider $\bigcup M(i)$; since each $M(i)$ is a sub-half-net of S, we have $\bigcup M(i) \subset S$. But, since $S = \bigcup (S \cdot J_i)$, then $S \subset \bigcup M(i)$, and we have S also represented in the form: $S = \bigcup M(i)$.

Suppose that there is a net R, such that $M \subset R \subset S = \bigcup M(i)$. Assume that it has already been shown that $M(n)$ is contained in R; we distinguish two cases:

(1) $n = 2k$. $M(2k) = (S \cdot J_{2k}) \cdot \bigcup_{i \geq k} K(i) \subset R$

$$\begin{aligned} M(2k+1) &= (S \cdot J_{2k+1}) \cdot \bigcup_{i \geq k+1} K(i) \\ &= (S \cdot J_{2k})_L \cdot \{(S \cdot J_{2k+1}) - (S \cdot J_{2k})_L\} \cdot \bigcup_{i \geq k+1} K(i) \\ &= (S \cdot J_{2k})_L \cup \bigcup_{i \geq k} K(i). \end{aligned}$$

Then $M(2k+1)$ contains, in addition to elements of $M(2k)$, only lines which are on at least one point of $(S \cdot J_{2k}) \subset M(2k) \subset R$. Since R is a subnet of N, clearly these lines must be in R. That is, $M(2k+1) \subset R$.

(2) $n = 2k+1$. $M(2k+1) = (S \cap J_{2k+1}) \cup \bigcup_{i \geq k+1} K(i) \subset R.$

$$M(2k+2) = (S \cap J_{2k+2}) \cup \bigcup_{i \geq k+1} K(i).$$

Since $(S \cap J_{2k+2}) = (S \cap J_{2k+1})_P$ (by Lemma (a)), $M(2k+2)$ contains, in addition to elements of $M(2k+1)$, only points which are on at least two lines of $(S \cap J_{2k+1}) \subset M(2k+1) \subset R$. Since R is a subnet of N, clearly these points must be in R. That is, $M(2k+2) \subset R$.

By induction, we have $\bigcup_{i \geq 0} M(i) = S \subset R \subset S$, and hence $R = S$; that is, S is generated by M.

Note that in the chain of $M(i)$'s, $M(2n+2)$ is not a complete P-extension of $M(2n+1)$, since the only points added in $M(2n+2)$ are those lying on two or more lines of $S \cap J_{2n+1}$, which is only a part of $M(2n+1)$; that is, in $M(2n+2)$, not every two lines of different pencils may meet. However, from the Corollary to Lemma 1.1, we have that a net is generated by its sub-half-net, if and only if, it is the join of half-nets in a maximal extension chain of which the sub-half-net is the first member. Hence S may be written in the form: $S = \bigcup_{i \geq 0} M_i$, where $M_0 = M$, and where M_{i+1} is a complete (L or P) extension of M_i.

We are now ready to prove Theorem 1.5.

Since N is freely generated by its sub-half-net J, then N may be represented as $\bigcup J^{(i)}$, where $J = J^{(0)}$, and, as is our usual notation, $J^{(i+1)}$ is a complete *open* (L or P) extension of $J^{(i)}$ (Corollary to Lemma 1.5.). By Lemma (c), the subnet S of N is generated by

$$M = (S \cap J^{(0)}) \cup \bigcup_{i \geq 0} K(i), \text{ where } K(i) = \{(S \cap J^{(2i+1)}) - (S \cup J^{(2i)})_L\}.$$

To show that S is free over M, we consider again the chain of $M(i)$'s, where, as before,

$$M(2n) = (S \cap J^{(2n)}) \cup \bigcup_{i \geq n} K(i); \quad M(2n+1) = (S \cap J^{(2n+1)}) \cup \bigcup_{i \geq n+1} K(i).$$

By the Corollary to Lemma (a), we have that $(S \cap J^{(2n)})_L$ is an open L-extension of $(S \cap J^{(2n)})$. Since $(S \cap J^{(2n)})_L$ and $\bigcup_{i \geq n} K(i)$ are totally disconnected (Lemma (b)), it is easily verified then, that

$$M(2n+1) = (S \cap J^{(2n)})_L \cup \bigcup_{i \geq n} K(i)$$

is an open L-extension of $M(2n)$.

Similarly, we consider

$$M(2n+1) = (S \cap J^{(2n+1)} \cap \bigcup_{i \geq n+1} K(i),$$

and

$$M(2n+2) = (S \cap J^{(2n+2)}) \cup \bigcup_{i \geq n+1} K(i).$$

By the Corollary to Lemma (a), then $(S \cap J^{(2n+2)})$ is an open P-extension of $(S \cap J^{(2n+1)})$, and, since $(S \cap J^{(2n+2)})$ is contained in $(S \cap J^{(2n+2)})_L$, we know that $(S \cap J^{(2n+2)})$ and $\bigcup_{i \geq n+1} K(i)$ are totally disconnected (Lemma (b)). Hence, as is easily verified, $M(2n+2)$ is an open P-extension of $M(2n+1)$.

Then, in the chain of $M(i)$'s, each $M(i+1)$ is an open extension of $M(i)$, and hence $S = \bigcup_{i \geq 0} M(i)$ is free over M (Lemma 1.5).

Now let H be the subnet of S which is generated by $H_0 = S \cap J$, and let F be the subnet of S which is generated by F_0, where

$$F_0 = \bigcup_{i=0}^{\infty} \{S \cap J^{(2i+1)} - (S \cap J^{(2i)})_L\}.$$

Let σ be a homomorphism of H, and τ a homomorphism of F into the same net R. Then σ induces a homomorphism σ_0 of H_0 and τ induces a homomorphism τ_0 of F_0 into R. Since H_0 and F_0 are totally disconnected (Lemma (b)), σ_0 and τ_0 define a homomorphism ϕ_0 of $H_0 \cup F_0$ into R, such that ϕ_0 and σ_0 coincide on H_0, and ϕ_0 and τ_0 coincide on F_0.

But S is free over and generated by $H_0 \cup F_0 = M$, and hence ϕ_0 may be extended to a homomorphism ϕ of S into R. Then ϕ induces a homomorphism ϕ_H of H; since ϕ_H and σ coincide on elements of H_0, and H is generated by H_0, then $\phi_H = \sigma$ (Theorem 1.1). Similarly, ϕ induces a homomorphism ϕ_F of F, and $\phi_F = \tau$. Hence ϕ is an extension of σ and τ. Furthermore, since $H_0 \cup F_0 \subset H \cup F \subset S$, and S is generated by $H_0 \cup F_0$, clearly S is generated by $H \cup F$. Hence we have shown that $S = H * F$.

We have then only to verify that H is free over $H_0 = S \cap J$ and that F is a free net having a free set of generators F_0. Let σ_0 be any homomorphism of H_0 into a net R. Then there exists at least one homomorphism τ_0 of F_0 into the same net R (for example, the homomorphism which maps F_0 into a null-net of R). Since S is freely generated by $H_0 \cup F_0 = M$, then the homomorphisms σ_0 and τ_0 may be extended simultaneously to a homomorphism ϕ of S into R. But ϕ induces a homomorphism ϕ_H of H into R and clearly ϕ_H is an extension of σ_0, which proves that H is free over H_0. The above argument is symmetric with respect to F and H, so that F is free over F_0. Then F is clearly a free net, since it has a free set of generators F_0 (by Lemma (b) F_0 consists of lines only).

COROLLARY. *A subnet of a free net is free.*

For, if N is a free net, then N has a free set of generators J, and by the Theorem, a subnet S is of the form $S = H * F$, where F is a free net, and where H is free over and generated by $S \cap J$. Since J is a set of lines, then $S \cap J$ consists entirely of lines, and hence H is a free net. But the free sum of free nets is free, so that S is a free net.

THEOREM 1.6. *N is a free net, if and only if, for every net M and homomorphism ϕ such that $M\phi = N$, there is a subnet R of M, such that ϕ induces an isomorphism of R upon N.*

Proof. Suppose that N is a free net; then N has a free set of generators J, consisting of lines only.

Let M be a net, ϕ a homomorphism, such that $M\phi = N$. Consider any line x in J; then there is at least one line y in M, such that $x = y\phi$. For each x in J, select one and only one y in M, with $y\phi = x$. Let R_0 be the sub-half-net of M consisting of these lines y.

Then, clearly, ϕ induces a homomorphism ϕ_0 of R_0 upon J which is $1-1$. Since J has no incidence relations, ϕ_0^{-1} is a homomorphism of J upon R_0, and hence ϕ_0 is an isomorphism of R_0 upon J.

Since N is free over J, ϕ_0^{-1} may be extended to a homomorphism μ of N upon $N\mu \subset M$; let $N\mu = R$. But ϕ induces a homomorphism ψ of R upon $R\psi$, and we have $R_0\psi = J \subset R\psi \subset N$; since J generates N, then $R\psi = N$.

Hence $N\mu\psi = N$, and $j\mu\psi = (j\mu)\psi = (j\phi_0^{-1})\psi = j\phi_0^{-1}\phi_0 = j$, for every $j \in J$. Then $\mu\psi = 1$ (Theorem 1.1). But, since every y in $R = N\mu$ is of the form $x\mu$ for some x in N, we have that, for all y in R, $y\psi\mu = (x\mu)\psi\mu = (x\mu\psi)\mu = x\mu = y$, and hence $\psi\mu = 1$.

That is, ψ and μ are reciprocal isomorphisms of R and N, and we have $R \sim N$.

Conversely, if N is a net having the property of the theorem, then, since N is the homomorphic image of a free net F (Theorem 1.4), we have N isomorphic to a subnet of F. But a subnet of a free net is free (Corollary to Theorem 1.5); hence N is a free net.

1.6. Existence of subnets on a countably infinite free set of generators. The following Lemma is needed in the proof of the next theorem:

LEMMA C. *If $J^{(0)} \subset J^{(1)} \subset \cdots \subset J^{(i)} \subset \cdots$ is a maximal open extension chain generating the net $N = \bigcup_{i \geq 0} J^{(i)}$, and if the sub-half-net H of N is closed in $J^{(2i+1)}$ (for some i), then H generates a subnet K of N, having the following properties:*

I. $K \cap J^{(j)} = H \cap J^{(j)}, \quad j \leq 2i+1.$

II. $(K \cap J^{(2j)})_L = K \cap J^{(2j+1)}, \quad j > i.$

Proof. Let $H = H_0 \subset H_1 \subset \cdots \subset H_i \subset \cdots$ be the maximal extension chain of H in N. Then $K = \bigcup_{i \geq 0} H_i$ is the subnet of N which is generated by H (Corollary to Lemma 1.1).

We now prove, by complete induction with respect to j, the validity of the following propositions:

(A) If x is in H_{j+1}, but not in H_j, then x is in $J^{(2i+j+1)}$, but not in $J^{(2i+j)}$, for all $j \geq 0$.

(B) H_{j+1} is closed in $J^{(2i+j+1)}$, for all $j \geq 0$.

(B) is needed only for the inductive proof of (A).

Since H is closed in $J^{(2i+1)}$, every point of H carries one line of each pencil in H, and hence $H = H_1$. Then for $j = 0$, proposition (A) is vacuously true, and, since $H = H_1$ is closed in $J^{(2i+1)}$, proposition (B) is valid.

Suppose that (A) and (B) hold for $j = k$; we distinguish two cases:

Case 1. $k = 2n$; then H_{2n+1} is closed in $J^{(2i+2n+1)}$, and clearly, $H_{2n+2} \subset J^{(2i+2n+2)}$.

If x is in H_{2n+2}, but not in H_{2n+1}, then x is a point on at least two lines of $H_{2n+1} \subset J^{(2i+2n+1)}$. If x were in $J^{(2i+2n+1)}$, then x would be in H_{2n+1}, since H_{2n+1} is closed in $J^{(2i+2n+1)}$. Hence x is in $J^{(2i+2n+2)}$, but not in $J^{(2i+2n+1)}$, and we have established the validity of (A) for $j = k+1 = 2n+1$.

Suppose that P is in H_{2n+2}, and P lies on h in $J^{(2i+2n+2)}$. Since $J^{(2i+2n+2)} = (J^{(2i+2n+1)})_P$, h is necessarily in $J^{(2i+2n+1)}$. Then if P is also in $H_{2n+1} \subset J^{(2i+2n+1)}$, h is in H_{2n+1} and hence in H_{2n+2}, by closure condition (a) for H_{2n+1}. If P is not in H_{2n+1} then P is on at least two lines h', h'', of H_{2n+1}. But, by property (A) just established, then one of these two lines must be h, since $J^{(2i+2n+2)}$ is an open extension. Hence h is in H_{2n+2}.

If k, k', are in H_{2n+2}, and if k, k', lie on Q in $J^{(2i+2n+2)}$, then Q is in H_{2n+2}, since in $H_{2n+2} = (H_{2n+1})_P \subset J^{(2i+2n+2)}$, every pair of lines of different pencils meet in a point.

Hence (B) is established for $j = k+1 = 2n+1$.

Case 2. $k = 2n+1$; then H_{2n+2} is closed in $J^{(2i+2n+2)}$, and clearly $H_{2n+3} \subset J^{(2i+2n+3)}$.

If x is in H_{2n+3}, but not in H_{2n+2}, then x is a line on at least one point of $H_{2n+2} \subset J^{(2i+2n+2)}$. If x were in $J^{(2i+2n+2)}$, then x would be in H_{2n+2}, since

H_{2n+2} is closed in $J^{(2i+2n+2)}$. Hence x is in $J^{(2i+2n+3)}$, but not in $J^{(2i+2n+2)}$, and we have established the validity of (A) for $j = k+1 = 2n+2$.

If P is in H_{2n+3}, and if P lies on k in $J^{(2i+2n+3)}$, then k is in $H_{2n+3} = (H_{2n+2})_L \subset J^{(2i+2n+3)}$. If h, h' are in H_{2n+3}, and if h, h', meet in a point Q in $J^{(2i+2n+3)}$, then, since $J^{(2i+2n+3)} = (J^{(2i+2n+2)})_L$, Q is necessarily in $J^{(2i+2n+2)}$. Hence, if both h, h', are also in H_{2n+2}, then Q is in H_{2n+2}, by closure condition (b) for H_{2n+2}. Suppose that one of the lines, say h, is not in H_{2n+2}; then by (A), h is in $J^{(2i+2n+3)}$, but not in $J^{(2i+2n+2)}$, and hence Q must be the one point on h in $H_{2n+3} \subset J^{(2i+2n+3)}$, since $J^{(2i+2n+3)}$ is an open extension. Hence H_{2n+3} is closed in $J^{(2i+2n+3)}$, and (B) is valid for $j = k+1 = 2n+2$. This completes the induction.

Now $H \cdot J^{(j)}$ is contained in $K \cdot J^{(j)}$, certainly. Furthermore, if x is in $K \cdot J^{(j)}$, for $j \leq 2i+1$, then x is in H; for, if x were *not* in H, then, for some $k \geq 1$, x would be in H_{k+1}, but not in H_k. But then, by proposition (A), x would be in $J^{(2i+k+1)}$, but not in $J^{(2i+k)}$, $(k \geq 1)$, which is impossible since x is in $K \cdot J^{(2i+1)} \subset J^{(2i+1)}$.

Hence $K \cdot J^{(j)}$ is contained in $H \cdot J^{(j)}$, for $j \leq 2i+1$, so that we have finally, $K \cdot J^{(j)} = H \cdot J^{(j)}$, for $j \leq 2i+1$, proving I of our lemma.

Furthermore $(K \cdot J^{(2j)})_L \subset K \cdot (J^{(2j)})_L = K \cdot J^{(2j+1)}$, since $K_L = K$. Consider an element x in $K \cdot J^{(2j+1)}$, for $j > i$; we have two possibilities:

(i) x is in $J^{(2j)}$; then x is in $K \cdot J^{(2j)} \subset (K \cdot J^{(2j)})_L$.

(ii) x is not in $J^{(2j)}$; then x is a line on P of $J^{(2j)}$ and on no other point in $J^{(2j+1)}$. x is not in H, since $H \subset J^{(2i+1)}$, and we have $j > i$. Then x is in H_{2k+1}, but not in H_{2k}, for some $k \geq 1$, and x carries a point Q from H_{2k}. But then x is in $J^{(2i+2k+1)}$, not in $J^{(2i+2k)}$, on Q of $J^{(2i+2k)}$, by proposition (A) of this proof. Since x is also in $J^{(2j+1)}$, but not in $J^{(2j)}$, we must have $2j = 2i + 2k$, and $Q = P$. Then $Q = P$ is in $K \cdot J^{(2j)}$ and x is in $(K \cdot J^{(2j)})_L$.

Hence in both cases we have $K \cdot J^{(2j+1)} \subset (K \cdot J^{(2j)})_L$, and finally, $(K \cdot J^{(2j)})_L = K \cdot J^{(2j+1)}$, for $j > i$, proving II of our lemma.

THEOREM 1.7. *If the non-null net N is freely generated by its sub-half-net $J \neq N$, then N contains a free net on a countably infinite free set of generators.*

Proof. N may be represented in the form: $N = \bigcup J^{(i)}$, where $J^{(0)} = J$, and $J^{(i+1)}$ is a complete open (L or P) extension of $J^{(i)}$.

In $J^{(2i-2)}$, for $i > 1$ and sufficiently large, there are at least two lines of each pencil and at least two points on each of these lines, since N is not a

null-net. Since $N \neq J$, we have $J^{(2i-1)} \neq J^{(2i-2)}$, (property (2^0) for maximal open extension chains). Then there exists a line x in $J^{(2i-1)}$, but not in $J^{(2i-2)}$, and a point P from $J^{(2i-2)}$ which does not lie on x. It is no loss of generality to assume that x is a t-line, t^*, and that $r(P)$ does not meet t^* in $J^{(2i-1)}$.

We define, then, for the net N, a "ladder," $P - t^*$, as follows: Let $P_0 = t^* \cdot r(P)$; suppose that we have already defined P_{4j} for $j \geq 0$: Then we define:

$$s_{4j+1} = s(P_{4j}); \qquad P_{4j+2} = s_{4j+1} \cdot t(P);$$
$$r_{4j+3} = r(P_{4j+2}); \qquad P_{4j+4} = r_{4j+3} \cdot t^*.$$

Note that $P_{4j+2} < t(P)$, and $P_{4j} < t^*$, for all j. See Fig. 1.

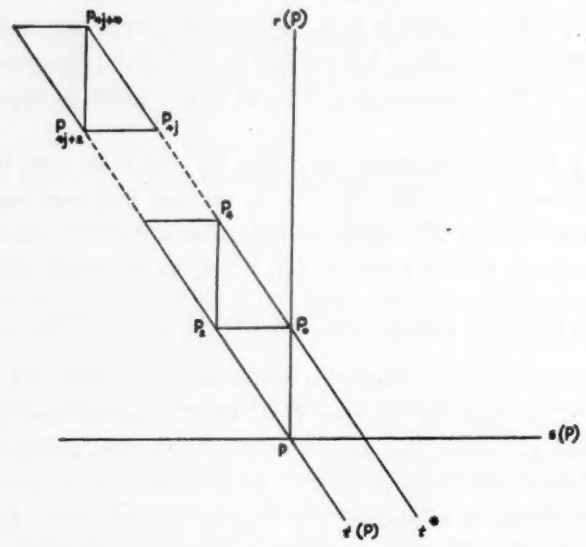

FIGURE 1.

Now we verify the validity of the following statements, for all $j \geq 0$:

P_{4j} is in $J^{(2i+4j)}$, but not in $J^{(2i+4j-1)}$;

s_{4j+1} is in $J^{(2i+4j+1)}$, but not in $J^{(2i+4j)}$;

P_{4j+2} is in $J^{(2i+4j+2)}$, but not in $J^{(2i+4j+1)}$;

r_{4j+3} is in $J^{(2i+4j+3)}$, but not in $J^{(2i+4j+2)}$.

$P_0 = t^* \cdot r(P)$ is in $J^{(2i)}$, but not in $J^{(2i-1)}$ (since t^* does not meet $r(P)$ in $J^{(2i-1)}$), and P_0 lacks an s-line.

Suppose that P_{4j} is in $J^{(2i+4j)}$, but not in $J^{(2i+4j-1)}$, and that P_{4j} lacks an s-line.

Then $s_{4j+1} = s(P_{4j})$ is in $J^{(2i+4j+1)}$, but not in $J^{(2i+4j)}$, and s_{4j+1} carries P_{4j} and no other point in $J^{(2i+4j+1)}$. Since $P_{4j} < t^*$, and, since P_{4j}, not being in $J^{(2i+4j-1)}$, is distinct from P, s_{4j+1} does not meet $t(P)$ in $J^{(2i+4j+1)}$.

Hence $P_{4j+2} = s_{4j+1} \cdot t(P)$ is in $J^{(2i+4j+2)}$, but not in $J^{(2i+4j+1)}$. Clearly, P_{4j+2} lacks an r-line.

Then $r_{4j+3} = r(P_{4j+2})$ is in $J^{(2i+4j+3)}$, but not in $J^{(2i+4j+2)}$. In $J^{(2i+4j+3)}$, r_{4j+3} carries P_{4j+2} and no other point. Since P_{4j+2} lies on $t(P)$ and hence on no other t-line of N, clearly r_{4j+3} does not meet t^* in $J^{(2i+4j+3)}$.

Then $P_{4j+4} = t^* \cdot r_{4j+3}$ is in $J^{(2i+4j+4)}$, but not in $J^{(2i+4j+3)}$, and P_{4j+4} lacks an s-line.

Hence the validity of the foregoing statements has been proven.

We shall show that the set of lines:

$$K = \{r(P), s(P), s_5, s_{13}, \cdots\} = r(P) \cup s(P) \cup \bigcup_{j=0}^{\infty} s_{8j+5}$$

is a free set of generators of a subnet S of N.

First, note that $r(P) \cup s(P) = K(0)$ generates the null-net $S(0)$ on P, and that $S(0)$ is free over $K(0)$. Since $r(P)$, $s(P)$, $t(P)$, are in $J^{(2i-1)}$, we have $S(0) \subset J^{(2i-1)}$. For definiteness, we assume that $S(0) \subset J^{(0)}$ since in later applications of this theorem this will be the case. This assumption does not constitute a restriction on the theorem, but is merely a matter of convenience.

Let $M(1) = S(0) \cup s_5$. Then $M(1)$ is closed in $J^{(2i+5)}$, since $S(0)$ is certainly closed in $J^{(2i+5)}$, and the only lines which meet s_5 in $J^{(2i+5)}$ are the lines t^* and r_3, neither of which is in $S(0)$. By Lemma C, then $M(1)$ generates a subnet $S(1)$ of N, which has the following properties:

I. $S(1) \cap J^{(j)} = M(1) \cap J^{(j)}$, $j \leq 2i + 5$

II. $(S(1) \cap J^{(2j)})_L = S(1) \cap J^{(2j+1)}$, $j > i + 2$.

From Theorem 1.5, then $S(1)$ is freely generated by

$$(S(1) \cap J^{(0)}) \cup \bigcup_{j=0}^{\infty} \{S(1) \cap J^{(2j+1)} - (S(1) \cap J^{(2j)})_L\}$$

$$= (S(1) \cap J^{(0)}) \cup \bigcup_{j=0}^{i+2} \{S(1) \cap J^{(2j+1)} - (S(1) \cap J^{(2j)})_L\} \cup$$

$$\bigcup_{j > i+2} \{S(1) \cap J^{(2j+1)} - (S(1) \cap J^{(2j)})_L\},$$

$$= (M(1) \cap J^{(0)}) \cup \bigcup_{j=0}^{i+2} \{M(1) \cap J^{(2j+1)} - (M(1) \cap J^{(2j)})_L\},$$

by I and II above.

But $M(1) = S(0) \cup s_5$, where $S(0) \subset J^{(0)}$ and s_5 is in $J^{(2i+5)}$, but not in $J^{(2i+4)}$. Hence

$M(1) \cap J^{(j)} = S(0) \cap J^{(j)} = S(0)$, for $j < 2i + 5$, and

$M(1) \cap J^{(2i+5)} = (S(0) \cap J^{(2i+5)}) \cup s_5 = S(0) \cup s_5$. Then

$$\bigcup_{j=0}^{i+1} \{M(1) \cap J^{(2j+1)} - (M(1) \cap J^{(2j)})_L\}$$
$$= \bigcup_{j=0}^{i+1} \{S(0) \cap J^{(2j+1)} - (S(0) \cap J^{(2j)})_L\}$$

is vacuous, but

$$\{M(1) \cap J^{(2j+5)} - (M(1) \cap J^{(2i+4)})_L\} = (S(0) \cup s_5) - S(0) = s_5.$$

Hence $S(1)$ is freely generated by $S(0) \cup s_5$.
Then $S(1)$ is certainly freely generated by

$$K(1) = r(P) \cup s(P) \cup s_5.$$

Note that, since $S(1) \cap J^{(j)} = M(1) \cap J^{(j)}$, $j \leq 2i + 5$, then t^* is not in $S(1)$, since t^* is in $J^{(2i-1)}$, but not in $M(1) = S(0) \cup s_5$.

Restating the above results in a form more suitable for induction purposes, we have that $S(1)$ is a subnet of N which has a free set of generators: $K(1) = r(P) \cup s(P) \cup s_5$, and which enjoys the following properties:

(1) $(S(1) \cap J^{(0)}) \cup \bigcup_{j=0}^{i+2} \{S(1) \cap J^{(2j+1)} - (S(1) \cap J^{(2j)})_L\} = S(0) \cup s_5$.

(2) $(S(1) \cap J^{(2j)})_L = S(1) \cap J^{(2j+1)}$, $j > i + 2$.

(3) t^* is not in $S(1)$.

Now assume that there exists a subnet $S(k)$ of N, with a free set of generators: $K(k) = r(P) \cup s(P) \cup \bigcup_{j=0}^{k-1} s_{8j+5}$, $(k \geq 1)$, enjoying the following properties:

(1) $(S(k) \cap J^{(0)}) \cup \bigcup_{j=0}^{i+4k-2} \{S(k) \cap J^{(2j+1)} - (S(k) \cap J^{(2j)})_L\}$

$$= S(0) \cup \bigcup_{j=0}^{k-1} s_{8j+5}.$$

(2) $(S(k) \cap J^{(2j)})_L = S(k) \cap J^{(2j+1)}$, $j > i + 4k - 2$.

(3) t^* is not in $S(k)$.

Suppose that s_{8k+5} is in $S(k)$; then, since s_{8k+5} is in $J^{(2i+8k+5)}$, it is in

$S(k) \cap J^{(2i+8k+5)} = (S(k) \cap J^{(2i+8k+4)})_L$. Furthermore, since s_{8k+5} is not in $J^{(2i+8k+4)}$, it is not in $S(k) \cap J^{(2i+8k+4)}$. Hence s_{8k+5} is on a point Q of $S(k) \cap J^{(2i+8k+4)}$. But s_{8k+5} carries P_{8k+4} and no other point of $J^{(2i+8k+4)}$. Hence $Q = P_{8k+4}$ is in $S(k)$; but P_{8k+4} lies on t^*, and hence t^* is in $S(k)$, which is impossible. Hence s_{8k+5} is not in $S(k)$.

Moreover, r_{8k+3} is not in $S(k)$, by the same line of reasoning as that used above.

Clearly $S(k) \cap J^{(2i+8k+5)}$ is closed in $J^{(2i+8k+5)}$, since $S(k)$ is a net. Then $M(k+1) = (S(k) \cap J^{(2i+8k+5)}) \cup s_{8k+5}$ is closed in $J^{(2i+8k+5)}$, since the only lines which meet s_{8k+5} in $J^{(2i+8k+5)}$ are t^* and r_{8k+3}, neither of which is in $S(k)$.

Then $M(k+1)$ generates a net $S(k+1) \subset N$, with properties:

I. $S(k+1) \cap J^{(j)} = M(k+1) \cap J^{(j)}$, $j \leq 2i + 8k + 5$.

II. $(S(k+1) \cap J^{(2j)})_L = S(k+1) \cap J^{(2j+1)}$, $j > i + 4k + 2$,

by Lemma C.

Then, by Theorem 1.5, $S(k+1)$ is freely generated by:

$$(S(k+1) \cap J^{(0)}) \cap \bigcup_{j=0}^{\infty} \{S(k+1) \cap J^{(2j+1)} - (S(k+1) \cap J^{(2j)})_L\}$$

$$= (S(k+1) \cap J^{(0)}) \cup \bigcup_{j=0}^{i+4k+2} \{S(k+1) \cap J^{(2j+1)} - (S(k+1) \cap J^{(2j)})_L\}$$

$$\cup \bigcup_{j > i+4k+2} \{S(k+1) \cap J^{(2j+1)} - (S(k+1) \cap J^{(2j)})_L\}$$

$$= (M(k+1) \cap J^{(0)}) \bigcup_{j=0}^{i+4k+2} \{M(k+1) \cap J^{(2j+1)} - (M(k+1) \cap J^{(2j)})_L\},$$

by I and II above

But $M(k+1) = (S(k) \cap J^{(2i+8k+5)}) \cup s_{8k+5}$, where s_{8k+5} is in $J^{(2i+8k+5)}$, but not in $J^{(2i+8k+4)}$, so that

$$M(k+1) \cap J^{(j)} = S(k) \cap J^{(j)} \text{ for } j < 2i + 8k + 5, \text{ and}$$

$$M(k+1) \cap J^{(2i+8k+5)} = (S(k) \cap J^{(2i+8k+5)}) \cup s_{8k+5}.$$

Since $(S(k) \cap J^{(2j)})_L = S(k) \cap J^{(2j+1)}$, for $j > i + 4k - 2$, we have that

$$\bigcup_{j=i+4k-1}^{i+4k+1} \{M(k+1) \cap J^{(2j+1)} - (M(k+1) \cap J^{(2j)})_L\}$$

$$= \bigcup_{j=i+4k-1}^{i+4k+1} \{S(k) \cap J^{(2j+1)} - (S(k) \cap J^{(2j)})_L\}$$

is vacuous.

However,

$$\{M(k+1) \cap J^{(2i+8k+5)} - (M(k+1) \cap J^{(2i+8k+4)})_L\}$$
$$= \{(S(k) \cap J^{(2i+8k+5)}) \cup s_{8k+5} - (S(k) \cap J^{(2i+8k+4)})_L\}$$
$$= \{(S(k) \cap J^{(2i+8k+5)}) \cup s_{8k+5} - (S(k) \cap J^{(2i+8k+5)})\} = s_{8k+5}.$$

Hence $S(k+1)$ is freely generated by

$$(S(k) \cap J^{(0)}) \cup \bigcup_{j=0}^{i+4k-2} \{S(k) \cap J^{(2j+1)} - (S(k) \cap J^{(2j)})_L\} \cup s_{8k+5}$$
$$= S(0) \cup \bigcup_{j=0}^{k-1} s_{8j+5} \cup s_{8k+5} = S(0) \cup \bigcup_{j=0}^{k} s_{8j+5}.$$

Then $S(k+1)$ is certainly freely generated by

$$K(k+1) = r(P) \cup s(P) \cup \bigcup_{j=0}^{k} s_{8j+5}.$$

Hence, we have an ascending chain of subnets:

$$S(0) \subset S(1) \subset \cdots \subset S(k) \subset S(k+1) \subset \cdots \subset N,$$

such that $S(i)$ is freely generated by $K(i) = r(P) \cup s(P) \cup \bigcup_{j=0}^{i-1} s_{8j+5}$ and $K(i) \subset K(i+1)$, for every i. Then it is easily verified that $S = \bigcup_{i=0}^{\infty} S(i)$ is a subnet of N which is freely generated by

$$K = \bigcup_{i=0}^{\infty} K(i) = r(P) \cup s(P) \cup \bigcup_{j=0}^{\infty} s_{8j+5}.$$

That is, S is a free subnet of N having a free set of generators, K, countably infinite in number.

1.7. Centered half-nets. We now consider a special class of half-nets which will find their direct counterpart in loop theory in the next section.

Definition. The half-net T is termed *P-centered,* for P a point in T, if the following conditions are satisfied:

(a) If Q is in T, then Q carries $r(Q)$, $s(Q)$, $t(Q)$, in T.

(b) $s(P)$ meets every x-line in T, for $x \neq s$.

$r(P)$ meets every x-line in T, for $x \neq r$.

Note that a net is P-centered for P, any point in the net.

THEOREM 1.8. *Given a net R and a sub-half-net M of R, containing th*

point P, then there exists a uniquely determined P-centered half-net, $M^* = (M, P)$, such that M^* is generated by M, and no proper sub-half-net of M^* is a P-centered half-net containing M.

Proof. There exist P-centered sub-half-nets of the net R; for instance, R itself. Consider the cross-cut, M^*, of all P-centered sub-half-nets of R which contain M. Now it is easily verified that M^* is a P-centered sub-half-net of R which contains M, and that there does not exist any proper P-centered sub-half-net of M^* containing M. Suppose that S is a sub-half-net such that $M \subset S \subset M^*$, with S closed in M^*. Then clearly, S, too, is P-centered, and hence $S = M^*$, so that M^* is generated by M and thus meets all requirements.

We shall use throughout the notation, (M, P), to represent the smallest P-centered half-net containing the half-net M, and generated by M.

COROLLARY. *If the net R is free over and generated by its sub-half-net J, and if J contains the point P, then R is freely generated by (J, P).*

This is an immediate consequence of Theorem 1. 8, and of Lemma 1. 3,(1).

Definition. If the net N is freely generated by its P-centered sub-half-net, (J, P), and if (J, P) has the following property:

(P_f) If Q is in (J, P), then Q is on $r(P)$ or on $s(P)$,

then we define (J, P) as a *P-centered free set of generators* of N. A *P-centered free net,* then, is a net having a P-centered free set of generators.

THEOREM 1. 9. *A net N has a P-centered free set of generators (for some point P in N) if, and only if, N has a free set of generators.*

Proof. If N has a P-centered free set of generators (J, P), then it is obvious that the sub-half-net K, consisting of all the r-lines in (J, P), together with the s-line, $s(P)$, constitutes a free set of generators of N.

Suppose that N has a free set of generators, M. Since M consists entirely of lines, in the maximal open extension chain of M, we have $M = M^{(1)}$, but $M^{(1)} \neq M^{(2)}$. Let P be any one of the points adjoined in $M^{(2)}$; then P lies on two and only two lines, say h, h', from M. Let M^* be the sub-half-net of N which is formed from M by deleting the lines h, h', and substituting, instead, the point $P = h \cdot h'$ in N. It is clear that N is freely generated by M^*, since $M^{(2)}$ is free over and generated by M^*. Then N is freely generated by (M^*, P), the P-centered sub-half-net of N which is determined by M^*, (Corollary to Theorem 1. 8). Furthermore, (M^*, P) is a P-centered free set of generators of N, since if there were a point Q in

(M^*, P), such that Q did not lie on $r(P)$ nor on $s(P)$, then Q would have three lines, one of each pencil, in (M^*, P). At most two of these lines would be lines of M, since M contains no points, and N is free over M. Then it is almost obvious, from the properties of P-centered half-nets, that there would be a P-centered half-net containing M and having but two of these lines, hence being properly contained in (M^*, P), which is impossible.

Definition. A net N containing the point P, is the *P-centered free sum* of its P-centered subnets, $(M(v), P)$, if the following conditions are satisfied:

(i) N is generated by $\bigcup_v (M(v), P)$.

(ii) If $\phi(v)$ are homomorphisms of $(M(v), P)$ into a net R, with $P\phi(v) = P\phi(w)$, for all v, w, then there exists a homomorphism ϕ of N, which coincides with $\phi(v)$ on $(M(v), P)$, for each v.

One verifies that if N is the P-centered free sum of its subnets, $(M(v), P)$, then $(M(v), P) \cap (M(w), P)$ consists of the null-net on P, for all $v \neq w$.

Note. If N is the P-centered free sum of its subnets, $(M(v), P)$, then N is the P-centered free sum of $(M(v), P)$ and the net generated by $\bigcup_{w \neq v} (M(w), P)$.

THEOREM 1.10. *If the net N containing the point P, is the free sum of its subnets, (K, P) and S, where (K, P) is generated by (K_0, P), and S is generated by S_0, then N is the P-centered free sum of the nets (K, P) and (T, P), where (T, P) is generated by (T_0, P), and where T_0 is the half-net which is obtained from S_0 by adjoining the point P.*

Proof. Let σ and τ be homomorphisms of (K, P) and (T, P), respectively, such that $P\sigma = P\tau$; clearly, τ induces a homomorphism ρ in S. There exists one and only one homomorphism ϕ of N which induces σ in (K, P) and ρ in S, since $N = (K, P) * S$. Thus ϕ coincides with the homomorphism τ of (T, P) in the point P and in S. But (T, P) is generated by $T_0 = S \cup P$ (since (T_0, P) is generated by T_0, and (T, P) is generated by (T_0, P)), implying that ϕ coincides with τ everywhere in (T, P), (Theorem 1.1), as was to be shown.

2. HALF-LOOPS AND CENTERED HALF-NETS.

2.1. Representation of half-loops by centered half-nets. A *half-loop* H is a set of elements with one composition, called addition, $a + b$, subject to the following rules:

(i) If $a+b=c$, and $a+b=d$, then $c=d$.

(ii) If $a+d=a+f$, then $d=f$; if $b+a=c+a$, then $b=c$.

(iii) There exists a null element, 0, satisfying

$$a+0=0+a=a, \text{ for every } a \text{ in } H.$$

The term, "half-loop," is not to be confused with "semi-loop," which is defined as follows: A *semi-loop, S,* is a half-loop which satisfies the additional condition:

(i'). If $a, b,$ are in S, then there is at least one element c in S, such that $a+b=c$.

A *loop, L,* is a semi-loop which satisfies the further condition:

(ii'). If $a, b,$ are in L, then there is at least one x, and at least one y in L, such that $a+x=b$, and $y+a=b$.

This definition is equivalent to the usual definition of a loop.[5]

With every half-loop T we associate an O-centered half-net, (M, O), as follows:

Let the points of (M, O) be ordered pairs of elements, (a, b), from T, subject to the following condition:

(P). (a, b) is a point in (M, O), if and only if, there is at least one element c in T, such that $a+b=c$.

Thus (o, o) is in (M, O), and we let $O=(o, o)$.

With every element b in T, we associate one and only one r-line, $r(b)$; one and only one s-line, $s(b)$; and one and only one t-line, $t(b)$. The point (x, y) in (M, O) is on $r(b)$, if and only if $x=b$; it is on $s(b)$, if and only if, $y=b$; and (x, y) of (M, O) is on $t(b)$, if and only if $x+y=b$.

In particular, if T is the null loop—i. e., if T consists of the null element alone, then (M, O) is the null net on O.

LEMMA 2.1. *(M, O), as defined above, is an O-centered half-net.*

Proof. We verify immediately, as direct consequences of the properties of the half-loop T and condition (P), that (M, O) is a half-net in which every point has one and only one line of each pencil. Also, $r(a)$ meets $s(O)$ in (a, o); $s(a)$ meets $r(O)$ in (o, a); and $t(a)$ meets $r(O)$ and $s(O)$ in (o, a) and (a, o), respectively, so that (M, O) satisfies the further conditions for an O-centered half-net.

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[5] See Baer (4).

We denote the dependence of (M, O) upon the half-loop T, by writing $(M, O) = N(T)$, where the letter "N" is used in the generic sense to signify half-net or net. In using this notation, we understand that the associated half-net is centered on the point $O = (o, o)$, and we reserve the letter "O" from now on for this role.

Now one verifies similarly, the

COROLLARY. *If T is a loop, then $N(T)$ is a net.*

This is a well-known theorem, of which our lemma is a generalization.[6]

LEMMA 2.2. *Given an O-centered half-net, (M, O), then there is a representation of (M, O) as $N(T)$, where T is a half-loop.*

Proof. To each point on the line, $s(O)$, we assign a symbol, letting "o" be the symbol for the point O. We wish to show that this set of symbols, T, forms a half-loop under addition as defined below:

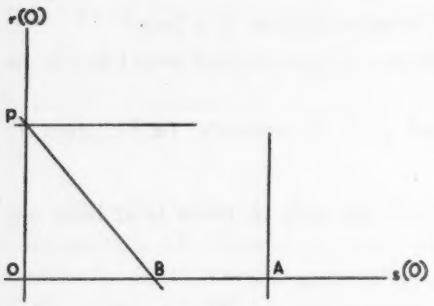

FIGURE 2.

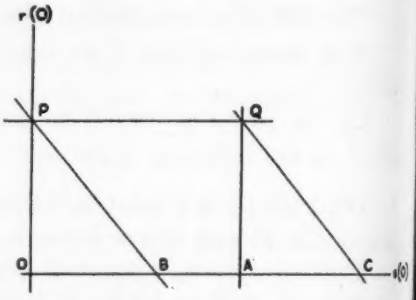

FIGURE 3.

Let A and B be points on $s(O)$ with symbols a and b, respectively. Then $t(B)$ and $r(A)$ are in (M, O), and $t(B)$ meets $r(O)$ in a point P. Furthermore, $s(P)$ is in (M, O). We proceed as follows:

(i) If $r(A)$ and $s(P)$ do not meet in (M, O), then we say there is no symbol in T representing $a + b$, or, simply that $a + b$ is not defined in T. (See Fig. 2.)

(ii) If $r(A)$ and $s(P)$ meet in Q, then $t(Q)$ is in (M, O), and $t(Q)$ meets $s(O)$ in a point C. Let the symbol for C be c. Then we define c to be $a + b$. (See Fig. 3.)

That the addition of symbols in T, as defined above, is single-valued,

[6] See Bol (1).

follows from the fact that each of the points and lines used to define $a+b$ is well-determined by the properties of (M, O). Hence, for elements of T, $a+b=c$, and $a+b=d$, imply that $c=d$. Clearly, the symbol "o" is the null element in this addition, and the two cancellation laws for T are easily verified. Hence T is a half-loop.

Returning to Fig. 3, if we assign to the point Q, the "coordinates" (a, b), we can show that the points of M may be mapped in a $1-1$ way upon the ordered pairs of symbols in T. Then points on the same r-line will have the same first coordinate, points on the same s-line will have their second coordinates the same, and points on the same t-line will be such that the sum of their two coordinates is the same for all points on the line. A point P will be in (M, O) and will have coordinates, (a, b), if and only if, $a+b$ is defined by the net addition described above. Hence we have (M, O) represented as $N(T)$, where T is a half-loop whose structure is completely determined by (M, O). We denote the dependence of the half-loop T upon the O-centered half-net, (M, O), by writing $T=L(M, O)$, where the letter "L" is used in the generic sense to indicate loop or half-loop.

One verifies, similarly, the following:

COROLLARY. *If (M, O) is a net, then $L(M, O)$ is a loop.*

This is a well-known theorem, of which our lemma is a generalization.[7]

The following relationships between half-loops and centered half-nets are easily verified, and are here given without proof, numbered for future reference.

1°. If K is a proper sub-half-loop of the half-loop L, then $N(K)$ is a proper sub-half-net of the half-net $N(L)$.

2°. If (M, O) is a proper sub-half-net of the half-net, (R, O), then $L(M, O)$ is a proper sub-half-loop of the half-loop, $L(R, O)$.

3°. If K and T are sub-half-loops of the half-loop L, then $N(K \cap T) = N(K) \cap N(T) \subset N(L)$.

2.2. Homomorphisms of half-loops and their associated half-nets.

Definition. A *homomorphism* η of a half-loop K, (into a half-loop, T), is a single-valued mapping of K which preserves sums—that is, if $a+b$ is defined in K, then so is $a\eta+b\eta$, and $(a+b)\eta=a\eta+b\eta$.

LEMMA 2.3. (a) *Given a homomorphism η of the half-loop K into the*

[7] See Bol (1).

half-loop T, then mapping (x, y) in $N(K) = (M, O)$ upon $(x, y)\phi = (x\eta, y\eta)$ in $N(T) = (R, O')$, constitutes a homomorphism of $N(K)$ into $N(T)$ with $O' = O\phi$.

(b) *Given a homomorphism ϕ of (M, O) into (R, O'), such that $O' = O\phi$, then ϕ is induced by a homomorphism η of $K = L(M, O)$ into $T = L(R, O')$ in the manner indicated in* (a).

Proof. (a) If (x, y) is in $N(K)$, then there is a z in K, such that $x + y = z$. Then $x\eta, y\eta$, are in T, with $x\eta + y\eta = (x + y)\eta = z\eta$ in T. Hence $(x\eta, y\eta)$ is a point in $N(T)$, and the mapping ϕ is a single-valued mapping of points of $N(K)$ upon points of $N(T)$. We have $O = (o, o)$ mapped under ϕ on $(o\eta, o\eta) = (o', o') = O'$, where o' is the null element in T.

Now (a, y) and (a, z) in $N(K)$ are mapped by ϕ upon $(a\eta, y\eta)$ and $(a\eta, z\eta)$, respectively. Hence points on the same r-line in $N(K)$ are mapped on points on the same r-line in $N(T)$. Similarly, (x, b) and (y, b) on the same s-line in $N(K)$ are mapped upon $(x\eta, b\eta)$ and $(y\eta, b\eta)$ on the same s-line in $N(T)$. Consider points (x, y) and $(x + y, o)$ on the same t-line in $N(K)$. $(x, y)\phi = (x\eta, y\eta)$, and $(x + y, o)\phi = [(x + y)\eta, o']$. But $(x + y)\eta = x\eta + y\eta$, and hence, $(x + y, o)\phi = (x\eta + y\eta, o')$. That is, points on the same t-line in $N(K)$ are mapped on points on the same t-line in $N(T)$.

Then ϕ is a homomorphism of $N(K)$ into $N(T)$, and we have proved (a).

(b) (M, O) determines $K = L(M, O)$, and hence we may write $(M, O) = N(K)$; similarly, (R, O') determines $T = L(R, O')$, and we have $(R, O') = N(T)$.

We define a single-valued mapping η as follows: $(x, o)\phi = (x\eta, o')$. Now (o, x) and (o, o) are on the same r-line in (M, O); then $(o, x)\phi$ and $(o, o)\phi = (o', o')$ must be on the same r-line in (R, O'). Hence $(o, x)\phi = (o', y)$, for some y in T.

(x, o) and (o, x) are on the same t-line in (M, O); then $(x, o)\phi = (x\eta, o')$ and $(o', x)\phi = (o', y)$ are on the same t-line in (R, O'). Then $x\eta + o' = o' + y$, or $y = x\eta$. Hence $(o, x)\phi = (o', x\eta)$.

Finally, we have $(x, y)\phi = (x\eta, y\eta)$, since $(x, y)\phi = (w, z)$ and $(x, o)\phi = (x\eta, o')$ on the same r-line in (R, O') implies that $w = x\eta$, and $(x, y)\phi$ and $(o, y)\phi = (o', y\eta)$ on the same s-line, implies that $z = y\eta$.

Now consider (x, y) and $(x + y, o)$ on the same t-line in (M, O); then $(x, y)\phi = (x\eta, y\eta)$, and $(x + y, o)\phi = [(x + y)\eta, o']$ must be on the same t-line in (R, O'). Hence $(x + y)\eta = x\eta + y\eta$; that is, η is a homomorphism of $K = L(M, O)$ into $T = L(R, O')$.

2.3. Concepts of freeness for half-loops and their associated half-nets.

A half-loop L is *free over* its sub-half-loop K if every homomorphism of K may be extended to a homomorphism of L.

Definition. The sub-half-loop H is *closed* in the half-loop K, if $H \subset K$, and if, given any two of the elements a, b, c, in H with $a + b = c$ in K, then the third element is in H.

The half-loop K is *generated* by its sub-half-loop S, if there is no proper sub-half-loop H of K which contains S and is closed in K.

It is easily verified that the half-loop K is generated by its sub-half-loop S, if and only if, $N(K)$ is generated by $N(S)$.

If the half-loop K is free over and generated by its sub-half-loop S, we shall say that K is *freely generated* by S.

LEMMA 2.4. *The half-loop T is freely generated by its sub-half-loop K, if and only if, $N(T)$ is freely generated by $N(S)$.*

Proof. Given a homomorphism ϕ of $N(K)$ into a net R, then R is 0ϕ-centered; let $O' = O\phi$, and let $H = L(R, O')$. Now H is a loop and there is a homomorphism η of K into H, which induces ϕ by Lemma 2.3 (b). Since the half-loop T is free over K, η may be extended to a homomorphism ρ of T into H. By Lemma 2.3, (a), then ρ induces a homomorphism ψ of $N(T)$ into $N(H) = R$, and ψ induces ϕ in $N(K)$. Hence $N(T)$ is free over $N(K)$. Since T is generated by K, then $N(T)$ is generated by $N(K)$, proving the Lemma in one direction.

Conversely, if $N(T)$ is freely generated by $N(K)$, by similar arguments we show that any homomorphism of K into a loop S may be extended to a homomorphism of T into S, so that T is free over K; and T is generated by K, since $N(T)$ is generated by $N(K)$.

Although, for the most part, we shall not be interested in an explicit representation for the elements of a loop in terms of its generators, some such representation is needed in the next section for our formulation of the Subloop Theorem. Accordingly, we define the following chain of half-loops:

Definition. If the loop R is generated by its sub-half-loop T, then R contains the following *generating chain* of half-loops:

$$T \subset T_1 \subset T_2 \subset T_3 \subset \cdots \subset T_i \subset \cdots \subset R,$$

where T_{i+1} is the set of all elements z, x, y, in R, such that: $z = a + b$, $x + a = b$, $a + y = b$, for a, b, in T_i. Furthermore, it is easily verified that $R = \bigcup_{i=0}^{\infty} T_i$.

Note. A direct development of the loop theory of Section **3** may be effected by a study of these generating chains and properly defined free generating chains for half-loops. In such a development, it is true, however, that some of the loop constructions analogous to our net constructions seem relatively more complicated.

The following lemma serves to make more specific the relationship between the elements in the generating chain of a loop R which is freely generated by its half-loop T, and our free net construction:

LEMMA 2.5. *If R is a loop which is freely generated by its half-loop T, and if $T \subset T_1 \subset T_2 \subset \cdots \subset T_i \subset \cdots \subset R$ is the generating chain of T in R, then the following conditions are satisfied, for every $i \geq 0$:*

(L_z). For every z in T_{i+1}, not in T_i, such that $z = a + b$ for a, b, elements in T_i, there exists an integer $j \geq 0$, such that $r(a)$, $s(b)$, are in $N(T)^{(2j+1)}$, but the point (a, b) is not in $N(T)^{(2j)}$.

(L_x). For every x in T_{i+1}, but not in T_i, such that $x + a = b$, for a, b, in T_i, there exists an integer $k \geq 0$ such that $s(a)$, $t(b)$, are in $N(T)^{(2k+1)}$, but the point (x, a) is not in $N(T)^{(2k+1)}$.

(L_y). For every y in T_{i+1}, but not in T_i, such that $a + y = b$, for a, b, in T_i, there exists an integer $l \geq 0$, such that $r(a)$, $t(b)$, are in $N(T)^{(2l+1)}$, but the point (a, y) is not in $N(T)^{(2l+1)}$.

Proof. Since R is freely generated by T, then $N(R) = \bigcup_{j=0}^{\infty} N(T)^{(j)}$ is freely generated by $N(T)$, and hence $N(T)^{(j+1)}$ is an open L or P extension of $N(T)^{(j)}$.

The conditions (L_z), (L_x), (L_y), are clearly satisfied for $i = 0$.

Assume that these conditions hold for all $i \leq M$. We prove, first, before proceeding with the induction, the validity of the following statement:

(K). Under the induction hypothesis, if c is any element in T_{i+1}, but not in T_i, $(i \leq M)$, then the following possibilities are exhaustive for the lines $r(c)$, $s(c)$, and $t(c)$ in $N(R)$:

I. There exists an integer $m \geq 1$, such that $r(c)$ is in $N(T)^{(2m+1)}$, but not in $N(T)^{(2m)}$, and such that there exists one and only one point in $N(T)^{(2m+1)}$ on $r(c)$, namely (c, u), with $u \neq c$, and u, $c + u$, in T_{i+1}.

II. There exists an integer $n \geq 1$, such that $s(c)$ is in $N(T)^{(2n+1)}$, but not in $N(T)^{(2n)}$, and such that there exists one and only one point in $N(T)^{(2n+1)}$ on $s(c)$, namely (v, c), with $v \neq c$, and v, $v + c$ in T_{i+1}.

III. There exists an integer $p \geq 1$, such that $t(c)$ is in $N(T)^{(2p+1)}$, but not in $N(T)^{(2p)}$, and such that there exists one and only one point in $N(T)^{(2p+1)}$ on $t(c)$, namely (w, w'), with $w, w', \neq c$, and $w, w', w + w'$ in T_{i+1}.

Proof of statement (K). Since c is in T_{i+1}, but not in T_i, there are three possibilities for c: (1) $c = w + w'$; w, w', in T_i; (2) $c + u = u'$; u, u', in T_i; (3) $v + c = v'$; v, v', in T_i. It is necessary to consider each of these possibilities separately:

Case (1). $c = w + w'$; since c is not in T_i, then $w, w' \neq c$. By the induction hypothesis, there exists an integer j, such that

$$r(w), s(w'), \varepsilon N(T)^{(2j+1)}, \text{ but } (w, w') \nmid N(T)^{(2j)}.$$

Then: $(w, w') = r(w) \cdot s(w')$ is in $N(T)^{(2j+2)}$, but not in $N(T)^{(2j+1)}$, and $r(w)$, $s(w')$, are the only lines on (w, w') in $N(T)^{(2j+2)}$.

$t(c) = t(w, w')$ is in $N(T)^{(2j+3)}$, but not in $N(T)^{(2j+2)}$, and (w, w') is the only point in $N(T)^{(2j+3)}$ on $t(c)$, exhibiting the possibility III of statement (K).

Since $(w, w') \neq (c, o)$ and $(w, w') \neq (o, c)$, we have

$$\begin{aligned} (c, o) &= t(c) \cdot s(o) \\ (o, c) &= t(c) \cdot r(o) \end{aligned} \quad \text{in } N(T)^{(2j+4)}, \text{ but not in } N(T)^{(2j+3)}.$$

Then $r(c) = r(c, o)$ is in $N(T)^{(2j+5)}$, but not in $N(T)^{(2j+4)}$, and (c, o) is the only point on $r(c)$ in $N(T)^{(2j+5)}$—that is, we have the situation in I of statement (K).

Also, $s(c) = s(o, c)$ is in $N(T)^{(2j+5)}$, but not in $N(T)^{(2j+4)}$, and (o, c) is the only point on $s(c)$ in $N(T)^{(2j+5)}$, (see II, under (K)).

Case (2). $c + u = u'$; since c is not in T_i, then $u, u', \neq c$.

By the induction hypothesis, there exists an integer k, such that $s(u)$, $t(u')$, are in $N(T)^{(2k+1)}$, but the point (c, u) is not in $N(T)^{(2k+1)}$.

Then $(c, u) = s(u) \cdot t(u')$ is in $N(T)^{(2k+2)}$, but not in $N(T)^{(2k+1)}$, and $s(u)$, $t(u')$, are the only lines on (c, u) in $N(T)^{(2k+2)}$.

$r(c) = r(c, u)$ is in $N(T)^{(2k+3)}$, but not in $N(T)^{(2k+2)}$, and (c, u) is the only point in $N(T)^{(2k+3)}$ on $r(c)$. (See I under (K)). $(c, u) \neq (c, o)$, since $u' \neq c$ implies that $u \neq o$.

Then $(c, o) = r(c) \cdot s(o)$ is in $N(T)^{(2k+4)}$, but not in $N(T)^{(2k+3)}$, and $r(c)$, $s(o)$, are the only lines on (c, o) in $N(T)^{(2k+4)}$.

$t(c)$ is in $N(T)^{(2k+5)}$, but not in $N(T)^{(2k+4)}$, and (c, o) is the only point in $N(T)^{(2k+5)}$ on $t(c)$. (See III under (K)). Since $c \neq o$, then $(o, c) \neq (c, o)$, and we have: $(o, c) = t(c) \cdot r(o)$ in $N(T)^{(2k+6)}$, but not in $N(T)^{(2k+5)}$, such that $t(c)$, $r(o)$, are the only lines in $N(T)^{(2k+6)}$ on (o, c).

$s(c) = s(o, c)$ is in $N(T)^{(2k+7)}$, but not in $N(T)^{(2k+6)}$, and (o, c) is the only point on $s(c)$ in $N(T)^{(2k+7)}$. (See II under (K)).

Case (3). $v + c = v'$; since c is not in T_i, then $v, v' \neq c$.

By the induction hypothesis, there exists an integer l, such that $r(v)$, $l(v')$, are in $N(T)^{(2l+1)}$, but the point (v, c) is not in $N(T)^{(2l+1)}$.

Then $(v, c) = r(v) \cdot t(v')$ is in $N(T)^{(2l+2)}$, but not in $N(T)^{(2l+1)}$, and $r(v)$, $t(v')$, are the only lines in $N(T)^{(2l+2)}$ on (v, c).

$s(c) = s(v, c)$ is in $N(T)^{(2l+3)}$, but not in $N(T)^{(2l+2)}$, and (v, c) is the only point in $N(T)^{(2l+3)}$ on $s(c)$. (See II under (K)).

$(v, c) \neq (o, c)$, since $v' \neq c$ implies that $v \neq o$.

Then $(o, c) = s(c) \cdot r(o)$ is in $N(T)^{(2l+4)}$, but not in $N(T)^{(2l+3)}$, and $s(c)$, $r(o)$, are the only lines in $N(T)^{(2l+4)}$ on (o, c).

$t(c) = t(o, c)$ is in $N(T)^{(2l+5)}$, but not in $N(T)^{(2l+4)}$, and (o, c) is the only point of $N(T)^{(2l+5)}$ which is on $t(c)$. (See III, (K)).

Since $c \neq o$, then $(o, c) \neq (c, o)$, and we have

$(c, o) = t(c) \cdot s(o)$ in $N(T)^{(2l+6)}$, but not in $N(T)^{(2l+5)}$, such that $t(c)$, $s(o)$, are the only lines in $N(T)^{(2l+6)}$ on (c, o).

$r(c) = r(c, o)$ is in $N(T)^{(2l+7)}$, but not in $N(T)^{(2l+6)}$, and (c, o) is the only point in $N(T)^{(2l+7)}$ on $r(c)$. (See I under (K)).

This completes the verification of statement (K), and we are ready to proceed with the inductive proof of the Lemma.

By the induction hypothesis, conditions (L_z), (L_x), (L_y), hold for all $i \leq M$. We now show that they hold for $i = M + 1$, which will complete the proof of the Lemma. We consider, first, condition (L_z): Given z, an element in T_{M+2}, but not in T_{M+1}, which is of the form $z = a + b$, where a, b, are elements of T_{M+1}, then we wish to show the existence of an integer j, such that $r(a)$, $s(b)$ are in $N(T)^{(2j+1)}$, but the point (a, b) is not in $N(T)^{(2j+1)}$. Since z is not in T_{M+1}, then $a, b, \neq z$, and $a, b, \neq o$. Not both a, b, are in T, since then $z = a + b$ is in $T_1 \leq T_{M+1}$.

First, suppose that both a and b are not in T; then

(1) a is in T_{i+1}, but not in T_i, for some $i \leq M$, and by statement (K), there exists an integer m, such that $r(a)$ is in $N(T)^{(2m+1)}$, but not in $N(T)^{(2m)}$, and such that there exists one and only one point in $N(T)$ on $r(a)$, namely (a, u), with $u \neq a$, and $u, a+u$ in T_{i+1}. Now $(a, u) \neq (a, b)$, since if $(a, u) = (a, b)$, then $z = a+b = a+u$ is in $T_{i+1} \subset T_{M+1}$, which is impossible. Hence $r(a)$ is in $N(T)^{(2m+1)}$, but not in $N(T)^{(2m)}$, and $P \neq (a, b)$ is the only point on $r(a)$ in $N(T)^{(2m+1)}$; that is, (a, b) is not in $N(T)^{(2m+1)}$.

(2) b is in T_{j+1}, but not in T_j, for some $j \leq M$, and by statement (K), there exists an integer n, such that $s(b)$ is in $N(T)^{(2n+1)}$, but not in $N(T)^{(2n)}$, and such that there exists one and only one point in $N(T)^{(2n+1)}$ on $s(b)$, namely (v, b), with $v \neq b$, and $v, v+b$, in T_{j+1}. Now $(v, b) \neq (a, b)$, since if $(v, b) = (a, b)$, then $z = a+b = v+b$ would be in $T_{j+1} \subset T_{M+1}$. Hence $s(b)$ is in $N(T)^{(2n+1)}$, but not in $N(T)^{(2n)}$, and $Q \neq (a, b)$ is the only point on $s(b)$ in $N(T)^{(2n+1)}$; that is, (a, b) is not in $N(T)^{(2n+1)}$.

Let $J = \max(m, n)$; then $r(a)$ and $s(b)$ are in $N(T)^{(2J+1)}$, but the point (a, b) is not in $N(T)^{(2J+1)}$, and our induction carries through in the case both a and b are not in T.

Suppose that a is in T, but b is not in T. Then, clearly, $r(a)$ is in $N(T) \subset N(T)^{(2n+1)}$, so that both $r(a)$, $s(b)$ are in $N(T)^{(2n+1)}$, but (a, b) is not in $N(T)^{(2n+1)}$.

Finally, if b is in T, but a is not in T, then $s(b)$ is in $N(T) \subset N(T)^{(2n+1)}$ so that both $r(a)$ and $s(b)$ are in $N(T)^{(2m+1)}$, but (a, b) is not in $N(T)^{(2m+1)}$.

Thus we have carried through the induction in showing that condition (L_z) also holds for $i = M+1$. The induction in proving the conditions (L_z) and (L_y), for $i = M+1$ goes through by similar arguments, and this concludes the proof of Lemma 2.5.

Definition. The loop L is the *free sum* of its subloops $L(v)$, (we write $L = \overset{*}{\Sigma} L(v)$), if the following conditions are satisfied:

(1) L is generated by $\bigcup L(v)$.

(2) If $\eta(v)$ are homomorphisms of the $L(v)$ into a loop K, then there is a homomorphism η of L into K which coincides with $\eta(v)$ in $L(v)$.

The following statement of the relationship between free sums of loops and free centered sums of nets is easily verified by means of preceding lemmas:

4°. The loop T is the free sum of its subloops, $T(v)$, if and only if, $N(T)$ is the O-centered free sum of its (O-centered) subnets, $N(T(v))$.

Definition. If the loop T is freely generated by its sub-half-loop K, and if K has the following property:

(f) If $a, b,$ are in K, then $a + b$ exists in K, if and only if, at least one of the elements $a, b,$ is o, then K is defined as a *free set of generators* of T. A *free loop,* then, is a loop which has a free set of generators.

Remark. This definition differs from the usual one (in group theory), in so far as we include o in the free set of generators, whereas o is usually excluded. This has technical advantages, but does not make any real difference.

Note that if $T = \overset{*}{\Sigma} T(v)$, and if the $T(v)$ are free loops, then T is free. The following statement is easily demonstrated:

5°. If the loop T has a free set of generators, K, then $N(T)$ has an O-centered free set of generators, $N(K)$, and conversely.

3. LOOPS.

We are now able to "translate" directly into loop theory the findings of preceding sections. At the same time, it will be of interest to note comparisons with classical results from the theory of free groups and free Abelian groups. That our definitions of free loop, free sums of loops, etc., may be formulated to embrace the usual definitions of free group, free Abelian group, free products of groups, and direct sums of Abelian groups, is exhibited in the Appendix. Hence we do have a basis for comparison of the two theories.

3. 1. Theorems on free loops and their generalizations. We have the following general existence theorem:

THEOREM 3. 1. (*General Existence Theorem*). *A half-loop H is contained in one, and essentially only one, loop T which is freely generated by H.*

Proof. By Theorem 1. 2, $N(H)$ may be embedded in one and essentially only one net M, which is freely generated by $N(H)$. Clearly, $M = (M, O)$. Let $T = L(M, O)$. Then T is a loop which is freely generated by H, (Lemma 2. 4). Furthermore, T is essentially uniquely determined, since the net construction is essentially unique.

The statement of Theorem 3. 1 is totally false for half-groups or Abelian half-groups,[8] since it is not true that every half-group (Abelian half-group)

[8] An (Abelian) half-group is a half-loop which is associative (and commutative). A semi-group is an associative semi-loop.

may be embedded in a group (Abelian group). This is seen in the following discussion.

Malcev [9] showed that, while a commutative semi-group [8] is always embeddable in a group, there exist non-commutative semi-groups which can not be embedded in a group. We use Malcev's necessary condition for the embedding of a semi-group in a group to exhibit an example of an Abelian half-group which is not embeddable in a group:

Example. The following condition is easily verified to be necessary for the embedding of an Abelian half-group into an Abelian semi-group:

(A) If $a, b, c, d, x, y, z, w,$ are any eight elements of an Abelian half-group satisfying the relations:

$$a+x=x+a=b+y=y+b; \qquad c+x=x+c=d+y=y+d;$$
$$a+z=z+a=b+w=w+b;$$

then, necessarily, $c+z=z+c=d+w=w+d.$

Consider, then, the Abelian half-group H consisting of the fourteen distinct elements: $o, a, b, c, d, x, y, z, w, p, q, r, s, t,$ with the following sums defined:

$o+e=e+o=e,$ for every e in H;

$a+x=x+a=b+y=y+b=p;\ c+x=x+c=d+y=y+d=q$

$a+z=z+a=b+w=w+b=r;\ c+z=z+c=s;$

$$d+w=w+d=t.$$

Then H does not satisfy condition (A) and, hence, cannot be embedded in an Abelian semi-group (or group).

COROLLARY. *The half-loop K is a loop, if and only if, K has the following property:*

(I) If $T, R,$ are loops generated by $K,$ then there exists an isomorphism of T upon R which leaves every element of K fixed.

Proof. If K is a loop, then K enjoys property (I) trivially.

Suppose that K is a half-loop, but not a loop. Then K may be embedded in one and essentially only one loop $T,$ which is freely generated by K (Theorem 3.1). By Theorem 1.3, $N(K)$ may be embedded in a net $M,$ which is generated by $N(K)$, but not free over $N(K)$. Let $R=L(M, O)$;

[9] See Malcev (1).

then R is a loop which is generated by K, but not free over K (Lemma 2.4). Now if there were an isomorphism between T and R, leaving every element of K fixed, then both T and R would be free over K, which is impossible. Hence K does not have property (I).

Note. This Corollary is false in the associative case, since Abelian semigroups may be embedded in one and essentially only one smallest group, which is necessarily Abelian.

Theorem 3.1 proves the existence of free loops, so that we have immediately the following theorem, which is true also for groups [10] and for Abelian groups.[11]

THEOREM 3.2. *Every loop is the homomorphic image of a free loop.*

This theorem follows from the net theorem, Theorem 1.4, or may be verified directly.

Definition. The subloop L' of the loop L is the *commutator-associator* [12] subloop of L, if it is generated by all the elements x in L which satisfy at least one of the following conditions:

(i) $(a+b)+x=b+a$ (iv) $[a+(b+c)]+x=(a+b)+$

(ii) $x+(a+b)=b+a$ (v) $x+[(a+b)+c]=a+(b+$

(iii) $[(a+b)+c]+x=a+(b+c)$ (vi) $x+[a+(b+c)]=(a+b)+$

for $a, b, c,$ elements of L.

LEMMA 3.1. *If S, T, are two free sets of generators of the free loop L, and if $n, m,$ are cardinal numbers representing the number of elements in S, T, respectively, then $n=m$.*

Note. We define, then, the *rank* of a free loop as the number of non-zero elements in a free set of generators of the loop.

Proof. Form the free Abelian group A on the generators S, and the free Abelian group B on the generators T.

Then the identity map of $S \subset L$ upon $S \subset A$ may be extended to a homomorphism η of L upon $L\eta = A$, since L is free over S. $K(\eta)$, the kernel of η, is the commutator-associator subloop L' of L. Hence $L/L' \sim A$.

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[10] See Kurosch (1).

[11] See Lefschetz (1), p. 50.

[12] The subloop L' has been introduced by Bruck ((2), pp. 267, 273) under the name of "the centrally derived loop of L."

Similarly, the identity map of $T \subset L$ may be extended to a homomorphism β of L upon $L\beta = B$, and $K(\beta) = L'$, so that $L/L' \sim B$. Hence $A \sim B$; but isomorphic free Abelian groups have the same rank,[13] and therefore, $n = m$.

COROLLARY. *If L, H, are free loops of rank n, m, respectively, then $L \sim H$, if and only if $n = m$.*

This is an immediate consequence of the Lemma.

THEOREM 3.3. *If the loop T is freely generated by its sub-half-loop $K \neq T$, then T contains a free subloop of infinite rank.*

Proof. $N(T)$ is free over and generated by its sub-half-net $N(K)$, and $N(K) \neq N(T)$ (Lemma 2.4, and property 1^0, **2**). By Theorem 1.7 $N(T)$ contains a free subnet S on a countably infinite free set of generators K. It is clear from the proof of Theorem 1.7 that we may require the point 0 to be in K. Then S has an O-centered free set of generators (K, O), and $L(K, O) = R$ is therefore a free subloop of T having a countably infinite free set of generators, $L(K, O)$.

COROLLARY. *A free loop $L \neq 0$ contains a free subloop of countably infinite rank.*

Note that, if L is a free loop of finite rank, it is clear that the set L is countable, since a finite number of elements is introduced at each of the countable number of stages in the construction of L. Therefore the statement that L contains a free subloop of countable rank is a "best possible result."

Remark. By a "ladder" construction similar to the one used in Theorem 1.7, we obtain the following set of elements G which constitute a free set of generators for a subloop of infinite rank of the free loop on one generator, a:

$$G = \{a + a,\ a + [a + (a + a)],\ a + (a + \{a + [a + (a + a)]\}), \cdots\}.$$

Corresponding to the above Corollary, we have the following theorems for groups and Abelian groups:

If H is a free group of rank greater than or equal to 2, then H contains a free subgroup of infinite rank.[14] If A is a free Abelian group, and if B is a subgroup of A, then the rank of B cannot exceed the rank of A.[15]

LEMMA 3.2. *If $L = J * K$, and if M is the cross-cut of all normal subloops*[16] *of L containing K, then $J \sim L/M$.*

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[13] See Lefschetz (1), p. 50.

[14] See Levi (1).

[15] See Lefschetz (1), p. 50.

[16] See Baer (4).

Proof. Let ϕ be the homomorphism of J into L such that $j\phi = j$, for every j in J, and let ψ be the homomorphism of K into L, such that $K\psi = 0$. Since $L = J * K$, then ϕ and ψ may be extended simultaneously to a homomorphism η of L into L, i. e., to an endomorphism of L. Then $J = L\eta$, and $j\eta = j$, for every j in J.

We verify that the kernel of η is M, as follows: Suppose that ρ is any homomorphism of L such that $K\rho = 0$; then $L\rho$ is generated by $J\rho$, and hence $L\rho = J\rho$. Then $L\rho = J\rho = L\eta\rho$, and $\rho = \eta\rho$, since $(j)\eta\rho = (j\eta)\rho = j\rho$, for every j in J, (see Appendix, Theorem 4. 1). This implies that the kernel of η is a part of the kernel of ρ. That is, the kernel of η is M. Hence we have $J \sim L/M$.

THEOREM 3. 4. (*Subloop Theorem*). *If the loop R is freely generated by its half-loop T, and if H is a subloop of R, then H is the free sum of loops G and F, where*

i. *G is freely generated by $H \cap T$.*

ii. *F is a free loop with a free set of generators*

$$F_0 = \bigcup_{i=0}^{\infty} \{(H \cap T_{i+1}) - (H \cap T_i)_1\},$$

where $T \subset T_1 \subset \cdots \subset T_i \subset \cdots \subset R$, is the generating chain of T in R.

Proof. $N(R)$ is freely generated by $N(T)$ (Lemma 2. 4). By Theorem 1. 5, the subnet $N(H)$ of $N(R)$ is the free sum of nets S and W, where S is a free net with a free set of generators

$$S_0 = \bigcup_{j=1}^{\infty} \{[N(H) \cap N(T)^{(2j+1)}] - [N(H) \cap N(T)^{(2j)}]_L\},$$

and W is the subnet of $N(H)$ which is freely generated by

$$N(H) \cap N(T) = N(H \cap T).$$

Now $N(H)$, $N(T)$, $N(H \cap T)$, are all O-centered; then $N(H)$ is the O-centered free sum of the nets W and S^*, where S^* is generated by $(S_0, 0)$, the smallest O-centered half-net containing S_0 and the point O (Theorem 1. 10).

Since $W = (W, O)$ is freely generated by $N(H \cap T)$, then $G = L(W, 0)$ is freely generated by $H \cap T$ (Lemma 2. 4).

S^* is a free net with an O-centered free set of generators $(S_0, 0)$ (Theorem 1. 9), and hence $F = L(S^*, O)$ is a free loop with a free set of generators, $L(S_0, O)$ (Property 5°, 2).

Hence $H = F * G$, and we have only to show that $F_0 = L(S_0, O)$:

Consider any element which is in F_0, and hence is in the half-loop, $(H \cap T_{i+1}) - (H \cap T_i)_1$, for some integer i; there are three possibilities to consider:

(1) z in $(H \cap T_{i+1}) - (H \cap T_i)_1$ (and hence z in T_{i+1}, but not in T_i), such that $z = a + b$; a, b, in T_i.

By Lemma 2.5, there is an integer $j \geq 0$, such that $r(a)$ and $s(b)$ are in $N(T)^{(2j+1)}$, and such that (a, b) is not in $N(T)^{(2j+1)}$. Then $(a, b) = r(a) \cdot s(b)$ is in $N(T)^{(2j+2)}$, but not in $N(T)^{(2j+1)}$, and $r(a)$, $s(b)$, are the only lines on (a, b) in $N(T)^{(2j+2)}$. Hence $t(z) = t(a, b)$ is in $N(T)^{(2j+3)}$, but not in $N(T)^{(2j+2)}$, and $t(z)$ carries (a, b) and no other point in $N(T)^{(2j+3)}$.

Since z is in $(H \cap T_{i+1}) - H \cap T_i)_1$, then z is in H, but neither a nor b is in H (for, if both a, b, were in H, then $z = a + b$ would be in $(H \cap T_i)_1$, which is impossible; and if one of the elements a, b were in H, then the other would also be in H, since z is in H).

Then $r(a)$ is not in $N(H)$, since, if it were, the point (a, o) on $r(a)$ and $s(o)$ would be in $N(H)$ and a would be in H. Similarly, the line $s(b)$ is not in $N(H)$. Consequently, the point (a, b) is not in $N(H)$. However, the line $t(z) = t(z, o)$ is in $N(H)$, since z is in H.

Thus we have that $t(z)$ is in

$$[N(H) \cap N(T)^{(2j+3)}] - [N(H) \cap N(T)^{(2j+2)}]_L,$$

and therefore, in S_0. Then $(z, o) = t(z) \cdot s(o)$ is in (S_0, O), and z is in $L(S_0, O)$.

(2) x is in $(H \cap T_{i+1}) - (H \cap T_i)_1$, such that $x + a = b$; a, b, in T_i.

By Lemma 2.5, there exists an integer $k \geq 0$, such that $s(a)$ and $t(b)$ are both in $N(T)^{(2k+1)}$, and such that the point (x, a) is not in $N(T)^{(2k+1)}$. Then $(x, a) = s(a) \cdot t(b)$ is in $N(T)^{(2k+2)}$, but not in $N(T)^{(2k+1)}$, and $s(a)$, $t(b)$, are the only lines on (x, a) in $N(T)^{(2k+2)}$. Hence $r(x, a) = r(x)$ is in $N(T)^{(2k+3)}$, but not in $N(T)^{(2k+2)}$, and the only point on $r(x)$ in $N(T)^{(2k+3)}$ is (x, a).

Since x is in $(H \cap T_{i+1}) - (H \cap T_i)_1$, then x is in H, but neither a nor b is in H. Then neither $s(a)$ nor $t(b)$ is in $N(H)$, although $r(x)$ is in $N(H)$. That is, $r(x)$ is in $[N(H) \cap N(T)^{(2k+3)}] - [N(H) \cap N(T)^{(2k+2)}]_L \subset S_0$. Hence $(x, o) = r(x) \cdot s(o)$ is in (S_0, O) and x is in $L(S_0, O)$.

(3) y in $(H \cap T_{i+1}) - (H \cap T_i)_1$, such that $a + y = b$, for a, b, in T_i.

By Lemma 2.5, there exists an integer $l \geq 0$, such that $r(a)$ and $t(b)$

are in $N(T)^{(2l+1)}$, and such that the point (a, y) is not in $N(T)^{(2l+1)}$. Then $(a, y) = r(a) \cdot t(b)$ is in $N(T)^{(2l+2)}$, but not in $N(T)^{(2l+1)}$, and $r(a)$, $t(b)$, are the only lines on (a, y) in $N(T)^{(2l+2)}$. Hence $s(y) = s(a, y)$ is in $N(T)^{(2l+3)}$, but not in $N(T)^{(2l+2)}$, and (a, y) is the only point on $s(y)$ in $N(T)^{(2l+3)}$.

Since y is in $(H \cap T_{i+1}) - (H \cap T_i)_1$, then y is in H, but neither a nor b is in H. Then neither $r(a)$ nor $t(b)$ is in $N(H)$, although $s(y)$ is in $N(H)$. That is, $s(y)$ is in $[N(H) \cap N(T)^{(2l+3)}] - [N(H) \cap N(T)^{(2l)}]_L \subset S_0$. Then $(o, y) = s(y) \cdot r(o)$ is in (S_0, O) and y is in $L(S_0, O)$.

This completes our proof showing that $F_0 \subset L(S_0, O)$.

We verify, next, that H is generated by

$$(H \cap T) \cup F_0 = (H \cap T) \cup \bigcup_{i=0}^{\infty} \{(H \cap T_{i+1}) - (H \cap T_i)_1\}.$$

For, we may represent H in the form: $H = \bigcup_{i=0}^{\infty} (H \cap T_i)$, since $H \subset R = \bigcup_{i=0}^{\infty} T_i$. But if we let $K(n) = (H \cap T_n) \cup \bigcup_{i=n}^{\infty} \{(H \cap T_{i+1}) - (H \cap T_i)_1\}$, then it is easily checked that H may also be represented in the form: $H = \bigcup_{i=0}^{\infty} K(i)$, and that H is generated by $K(o) = (H \cap T) \cup F_0$.

There exists one and only one endomorphism ϵ of H which leaves invariant every element in $(H \cap T) \cup F_0$, and which annihilates every element in $(L(S_0, O) - F_0)$ (since L is freely generated by $(H \cap T)$ and $L(S_0, O)$; and F_0 is a subset of the free set of generators, $L(S_0, O)$). But $\epsilon = 1$, since H is generated by $(H \cap T) \cup F_0$. Hence $(L(S_0, O) - F_0)$ is vacuous; i.e., $L(S_0, O) = F_0$.

We incorporate the general results of Theorem 3. 4, together with some obvious inferences of this theorem into the following theorem for later convenience:

General Subloop Theorem. If the loop R is freely generated by its half-loop T, if H is a subloop of R, and if G is the subloop of H which is (freely) generated by $H \cap T$, then $H = F * G$, where:

i. *F is a free loop whose rank is uniquely determined by H and T.*

ii. *F posseses a free basis F_0, such that the following property is true:*

(α) If α is an automorphism of R such that $H\alpha = H$ and $T\alpha = T$, then $F\alpha = F$ and α effects a permutation of the elements of F_0.

Property i. of F is an immediate consequence of Lemma 3. 2, and

property ii. is obviously true of the particular choice of the free loop F and basis F_0 in Theorem 3.4.

COROLLARY. *A subloop of a free loop is free.*

Proof. If the loop R is free, then R has a free set of generators, J. If H is a subloop of R, then, by the theorem above, $H = F * G$, where F is a free loop and where G is the subloop of H that is freely generated by $H \cap J$. Since J is a free set of generators, then it is obvious that $H \cap J$ forms a free set of generators of G, and hence that G is a free loop. But the free sum of free loops is free; hence H is a free loop.

The corresponding statements to the Corollary for groups,[17] and for Abelian groups [18] are true.

THEOREM 3.5. *K is a free loop, if and only if, for every loop H and homomorphism ϕ such that $H\phi = K$, there exists a (free) subloop S of H, such that ϕ induces an isomorphism of S upon K.*

This theorem may be deduced readily from Theorem 1.6 by the customary arguments involving net representations or may be verified directly by a proof similar to the one of Theorem 1.6.

Theorem 3.5 is true also for groups [19] and for Abelian groups.[20]

3.2. Theorems on free sums and generalized free sums of loops. We prove first the following lemma:

LEMMA 3.3. *Given $\{L(v)\}$, a set of loops having the same o-element, then $S = \bigcup L(v)$ is a half-loop, if S has the property:*

(A) *If two of the elements a, b, c, are in $L(v) \cap L(w)$, then $a + b = c$ in $L(v)$, if and only if $a + b = c$ in $L(w)$.*

Proof. The expression $a + b = c$, for a, b, c, in S, has meaning only if a, b, c, are all in some $L(v)$, with addition as defined in that $L(v)$. Hence if $a + b = c$, and $a + b = d$, in S, then $c = d$, by (A); and if $a + f = a + g$ in S, then $f = g$; or if $h + a = k + a$ in S, then $h = k$. Hence S is a half-loop.

We now generalize the concept of "free sum" in the following way:

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[17] See Schreier (1).
[18] See Lefschetz (1), p. 50.
[19] See Baer (5).
[20] See Eilenberg-MacLane (1).

Definition. The loop L is the *generalized free sum* of its subloops $L(v)$, if the following conditions are satisfied:

(a) L is generated by $S = \bigcup L(v)$.

(b) If $\alpha(v)$ are homomorphisms of the $L(v)$, with the property:

(*) If x is in $L(v) \cap L(w)$, then $x\alpha(v) = x\alpha(w)$,

then there exists a homomorphism α of L, which induces $\alpha(v)$ in $L(v)$.

Note. If x in $L(v) \cap L(w)$, $v \neq w$, implies that $x = 0$, then L is simply the *free sum* of its subloops $L(v)$.

Remark. If the half-loop H is the join of its sub-half-loops, $H(v)$, then every homomorphism α of H induces well-determined homomorphisms $\alpha(v)$ of $H(v)$ with the property

(*) If x is in $H(v) \cap H(w)$, then $x\alpha(v) = x\alpha(w)$.

If, conversely, there is given for every v, a homomorphism $\alpha(v)$ of $H(v)$, such that (*) is valid, then there exists one and only one homomorphism α of H inducing $\alpha(v)$ in $H(v)$. (Note that this last statement is no longer true if we have, instead, that H is generated by the $H(v)$, since the existence of the homomorphism of H would imply freeness.)

By virtue of the statements in the above remark, we may give the equivalent definition:

Definition. The loop L is the *generalized free sum* of its subloops $L(v)$, if and only if, L is freely generated by the join of the $L(v)$.

Note. If L is the (generalized) free sum of its subloops $L(v)$, then L is the (generalized) free sum of $L(w)$ and T, where T is the (generalized) free sum of the loops $L(v)$, for $v \neq w$.

If $\{L(v)\}$ is a set of loops, having the same o-element, and satisfying condition (A) of Lemma 3.3, then it is clear that L, the generalized free sum of the $L(v)$ exists. For, $S = \bigcup L(v)$ is a half-loop and consequently may be embedded in a loop L which is freely generated by S (Theorem 3.1).

Suppose that there is given a set of loops $\{L(v)\}$, and that there is given for every pair of indices $v \neq w$ an isomorphic correspondence $\alpha(v, w)$ mapping a certain subloop $L(v; w)$ of $L(v)$ upon a subloop $L(v; w)\alpha(v, w)$ of $L(w)$; these isomorphisms are subject to the following requirements:

(a) $\alpha(v, w) = \alpha(w, v)^{-1}$, so that $L(v; w)\alpha(v, w) = L(w; v)$.

(b) If x is an element in $L(v; w)$, and if $x\alpha(v, w)$ belongs to $L(w; z)$,

(so that $x\alpha(v,w)$ is in $L(w;v) \cap L(w;z)$), then x is in $L(v;z)$ and $x\alpha(v,z) = x\alpha(v,w)\alpha(w,z)$.

Then it is possible to construct a half-loop H which is the join of the $L(v)$ with the element x in $L(v;w)$ identified with the element $x\alpha(v,w)$ in $L(w;v)$, because now condition (A) of Lemma 3.3 is satisfied. The loop L which is freely generated by this half-loop H is called the *free sum of the $L(v)$ with amalgamated subloops.*

We use the terms "generalized free sum" and "free sum with amalgamated subloops" interchangeably—it being understood that the identification of corresponding elements has already been effected.

The existence of the generalized free sum (product) of groups, in the sense of our definition, seems to be an open question. Schreier [21] has proved the existence of the free sum of groups $G(v)$ with amalgamated subgroups $H(v)$, for the special case in which each $H(v)$ is isomorphic to a given group H. The existence of direct sums of Abelian groups with amalgamated subgroups also appears to be not a trivial question.

We shall use the notation: $L = \overset{*}{\sum} L(v)$ to denote the *generalized* free sum of the $L(v)$.

THEOREM 3.6. *(Subloop Theorem for Subloops of a (Generalized) Free Sum) If $L = \overset{*}{\sum} L(v)$ and if H is a subloop of L, then $H = F * \overset{*}{\sum} [H \cap L(v)]$. where F is a free loop such that:*

(i) *The rank of F is uniquely determined by the $L(v)$ and H.*

(ii) *There exists a free basis F_0 of F such that if α is an automorphism of L with $L(v)\alpha = L(v)$, for every v, and $H\alpha = H$, then $F\alpha = F$ and α effects a permutation of the elements in F_0.*

Proof. Since L is the generalized free sum of its subloops $L(v)$, then L is freely generated by its sub-half-loop $S = \bigcup L(v)$ (where suitable identification of elements in the $L(v)$ have been made). By the General Subloop Theorem, then H is the free sum of loops F and G, where G is the subloop of H which is freely generated by $H \cap S$, and where F is a free loop with the following properties:

(i) The rank of F is uniquely determined by H and S, and therefore, by H and the $L(v)$.

(ii) There exists a free basis F_0 of F, such that if α is an automorphism of L with $H\alpha = H$ and $S\alpha = S$, then $F\alpha = F$, and α effects a permutation of

[21] See Schreier (1).

the elements of F_0. Now if $L(v)\alpha = L(v)$, then certainly $S\alpha = S = \bigcup L(v)$, so that we have property (ii) as stated in this theorem.

Since $H \cap S = H \cap \bigcup L(v) = \bigcup [H \cap L(v)]$ and G is freely generated by $H \cap S$, then G is the generalized free sum of the $H \cap L(v)$, which concludes the proof.

COROLLARY. *If* $L = \overset{*}{\sum} L(v)$, *and if* H *is a subloop of* L, *then* $H = F * \overset{*}{\sum} [H \cap L(v)]$.

The corresponding theorem for subgroups of a free product of groups is as follows: [22]

If the group G is the free product of its subgroups $G(v)$, and if U is a subgroup of G, then $U = F * \overset{*}{\prod} U(i, v)$, where

i. F is a free group, and $U(i, v) = U \cap u_i G(v) u_i^{-1}$, with $v = v(i)$ and u_i in G.

ii. There is one and only one $U(j, v)$ in U which is conjugate to $U \cap wG(v)w^{-1} \neq 1$, for arbitrary v, w.

That our Corollary is not true for groups is exhibited in the following example:

Example. Consider $G = G_1 * G_2$, where G_1 is a cyclic group generated by a, and G_2 is a finite cyclic group generated by b. Then H, the subgroup of G which is generated by $a^{-1}ba$, does not have the structure required in the Corollary, since H is not free, although its crosscuts with G_1 and G_2 are zero.

There is no corresponding statement to the Corollary which can be made for subgroups of a direct sum of Abelian groups. For example, there exist Abelian groups without elements of finite order (except o), which are not direct sums of groups of rank 1.[23] Such a group, however, is a subgroup of an Abelian group which is the direct sum of groups, each of rank 1.

THEOREM 3.7. *If* $L = \overset{*}{\sum} L_i = \overset{*}{\sum} K_j$ *are two decompositions of the loop* L *into a free sum of loops, then there exist free loops* F_i, G_j, *such that*

$$L_i = F_i * \overset{*}{\sum_j} (L_i \cap K_j); \qquad K_j = G_j * \overset{*}{\sum_i} (L_i \cap K_j),$$

and

$$L = \overset{*}{\sum} F_i * \overset{*}{\sum_{i,j}} (L_i \cap K_j) = \overset{*}{\sum} G_j * \overset{*}{\sum_{i,j}} (L_i \cap K_j),$$

where $\overset{*}{\sum} F_i$ *and* $\overset{*}{\sum} G_j$ *are isomorphic free loops.*

[22] See Baer and Levi (1) and Takahasi (1).

[23] See Baer (2)

Proof. This theorem is an immediate consequence of Theorem 3.4 and Lemma 3.2.

The corresponding theorem for the decomposition of groups into free products is the following:[24]

If $G = \overset{*}{\prod} A_v = \overset{*}{\prod} B_w$ are two decompositions of the group G into a free product of groups, then there are free groups F_v and G_w, such that

$$A_v = \overset{*}{\prod_k} A_{vk} * F_v \text{ and } B_w = \overset{*}{\prod_i} B_{wi} * G_w, \text{ and}$$

$$G = \overset{*}{\prod_{v,k}} A_{vk} * \overset{*}{\prod} F_v = \overset{*}{\prod_{w,i}} B_{wi} * \overset{*}{\prod} G_w,$$

where A_{vk} and B_{wi} are pairwise conjugate and $\overset{*}{\prod} F_v$, $\overset{*}{\prod} G_w$ are isomorphic free groups.

There is no corresponding theorem for direct sums of Abelian groups.[25]

APPENDIX.

We consider a class $\mathfrak{R}$ of half-loops meeting the following requirements:

(I) If S is a sub-half-loop of the half-loop H in $\mathfrak{R}$, then S is in $\mathfrak{R}$.

(II) If the half-loop J is the homomorphic image of the half-loop H in $\mathfrak{R}$, then J is in $\mathfrak{R}$.

It is clear that the following three classes meet requirements (I) and (II) for a class $\mathfrak{R}$: the class $\mathfrak{E}$ of all half-loops; the class $\mathfrak{G}$ of all associative half-loops (i.e. half-groups); and the class $\mathfrak{A}$ of all associative and commutative half-loops, (i.e. Abelian half-groups). While these three classes by no means exhaust the possibilities, they are the only ones with which we shall here be concerned.

Definition. The loop L in $\mathfrak{R}$ is generated by its sub-half-loop J if no proper subloop of L contains J.

THEOREM 4.1. *If the loop L in $\mathfrak{R}$ is generated by its sub-half-loop J, and if ρ, σ, are homomorphisms of L into the loop S in $\mathfrak{R}$, such that $j\rho = j\sigma$, for every j in J, then $\rho = \sigma$.*

Proof. Let T be the set of elements t in L, such that $t\rho = t\sigma$, for every

[24] See Baer and Levi (1) and Takahasi (1).

[25] See Baer (1).

t in T. Clearly T contains J, and, since T is a sub-half-loop of L, T is in $\mathfrak{R}$. But then it is easily verified that T is, moreover, a subloop of L, and hence, since L is generated by J, we have $T = L$. That is $\rho = \sigma$.

Definition. The half-loop L in $\mathfrak{R}$ is *$\mathfrak{R}$-free over its sub-half-loop J* if it meets the following requirement:

(F) If σ is a homomorphism of J into the loop S in $\mathfrak{R}$, then there exists a homomorphism τ of L into the same loop S, which coincides with σ in J. (Note that $J\sigma$ and $L\tau$ are necessarily in $\mathfrak{R}$.)

THEOREM 4.2. *Given a half-loop J in $\mathfrak{R}$, then there is essentially at most one loop L in $\mathfrak{R}$, which is $\mathfrak{R}$-freely generated by J.*

Proof. Suppose that L and K are loops in $\mathfrak{R}$, each of which is $\mathfrak{R}$-freely generated by J. Since L is $\mathfrak{R}$-free over J, the identity map of $J \subset L$ upon $J \subset K$ may be extended to a homomorphism η of L into K; since K is generated by J, η is a homomorphism of L upon K, and we have $L\eta = K$. But we may extend, similarly, the identity map of $J \subset K$ upon $J \subset L$ to a homomorphism τ of K upon L, obtaining, then $K\tau = L$. Hence $L\eta\tau = L$, and $j\eta\tau = j$, for every j in J. By Theorem 4.1, then $\eta\tau = 1$. Similarly, $\tau\eta = 1$. Hence η and τ are reciprocal isomorphisms of L and K, and we have $L \sim K$.

Of particular interest are the $\mathfrak{R}$-free loops and the $\mathfrak{R}$-free sums of loops, which we define now:

Definition. If the loop L in $\mathfrak{R}$ is $\mathfrak{R}$-free over and generated by its sub-half-loop J, and if $x + y = z$ in J if and only if x or y is o, then we define J to be a *$\mathfrak{R}$-free set of generators* of L. A loop L in $\mathfrak{R}$ is *$\mathfrak{R}$-free* if it possesses a $\mathfrak{R}$-free set of generators.

Note that if J is a $\mathfrak{R}$-free set of generators of the $\mathfrak{R}$-free loop L, then every single-valued mapping η of J into the loop M in $\mathfrak{R}$, which maps the null element of J upon the null element of M, may be extended to a homomorphism of L into M.

Definition. The loop L in $\mathfrak{R}$ is the *generalized $\mathfrak{R}$-free sum* of its subloops $L(i)$, if it meets the following requirements:

(i) L is generated by $\bigcup L(i)$.

(ii) If there is given, for every i, a homomorphism $\alpha(i)$ of $L(i)$ (into the loop S in $\mathfrak{R}$), such that $x\alpha(i) = x\alpha(j)$, for every x in $L(i) \cap L(j)$,

then there exists a homomorphism α of L into S which coincides with $\alpha(i)$ in $L(i)$.

Note. Conditions (i) and (ii) are equivalent to requiring that L be $\mathfrak{R}$-freely generated by the set-theoretical sum of its subloops $L(i)$, which sum is a sub-half-loop of L.

If $L(i) \cap L(j) = o$, for every $i \neq j$, and if L meets (i) and (ii) above, then L is termed the $\mathfrak{R}$-*free sum* of its subloops $L(i)$.

Note, in particular, that a $\mathfrak{R}$-free loop is the $\mathfrak{R}$-free sum of $\mathfrak{R}$-free cyclic loops.

Now it is easy to verify that an $\mathfrak{A}$-free loop is a free Abelian group, a $\mathfrak{G}$-free loop is a free group, an $\mathfrak{A}$-free sum of loops is a direct sum of Abelian groups, and a $\mathfrak{G}$-free sum is a free sum of groups. For example, suppose that L is an $\mathfrak{A}$-free loop; then L has an $\mathfrak{A}$-free set of generators J. Construct the free Abelian group L^* on the set of generators J. Then every element in L^* has a unique representation in terms of the elements of J, and hence, given a single-valued map ρ of J into an Abelian group T, ρ may be extended to a homomorphism σ of L^* into T in the usual manner (i. e., for x in L^*, then $x = \Sigma n_i j_i$, where $j_i \in J$, and the n_i are integers; define $x\sigma = \Sigma n_i (j_i)\rho$). Hence L^* is an $\mathfrak{A}$-free loop with J as an $\mathfrak{A}$-free set of generators. But then $L^* \sim L$, since a $\mathfrak{R}$-free set of generators essentially uniquely determines a $\mathfrak{R}$-free loop (Theorem 4. 2).

UNIVERSITY OF ILLINOIS,
URBANA, ILLINOIS.

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ON THE CONVERGENCE AND DIVERGENCE OF CONTINUED FRACTIONS.*

By W. T. Scott and H. S. Wall.

1. Introduction. A sequence $x_1, x_2, x_3, \cdots$ of complex numbers is said to satisfy *condition* (H) if at least one of the following statements holds.

(a) The series $\Sigma \mid x_{2p+1} \mid$ diverges.

(b) The series $\Sigma \mid x_{2p+1}(x_2 + x_4 + \cdots + x_{2p})^2 \mid$ diverges.

(c) $\operatorname{Lim}_{p=\infty} \mid x_2 + x_4 + \cdots + x_{2p} \mid = \infty$.

This condition was first used in connection with continued fractions by Hamburger [2]. In this paper we show that a necessary condition for the continued fraction $\overset{\infty}{K}_{p=1} (1/b_p)$ to converge is that the sequence $\{b_p\}$ of its partial denominators satisfy condition (H). In particular, the continued fraction diverges if the series $\Sigma \mid b_p \mid$ converges (Stern [6], von Koch [4]), or, more generally, if the two series Σb_{2p} and Σb_{2p+1} converge, at least one absolutely. We show further that if $b_{2p-1} = k_{2p-1} z_p$, $b_{2p} = k_{2p}$, $k_1 > 0$, $k_{2p-1} \geq 0$, $R(k_{2p}) \geq 0$, $R(z_p) \geq \delta$, $\mid z_p \mid \leq M$, $p = 1, 2, 3, \cdots$, where δ and M are positive constants, then the continued fraction converges if and only if the sequence $\{k_p\}$ satisfies condition (H). This result includes and correlates convergence theorems of Stieltjes [7], Van Vleck [10], Hamburger [2], and Mall [5].

2. Necessity of condition (H). We shall now prove the following theorem.

Theorem A. *A necessary condition for the continued fraction* $\overset{\infty}{K}_{p=1} (1/b_p)$ *to converge is that the sequence* $\{b_p\}$ *of its partial denominators satisfy condition* (H).

Proof. We suppose that the sequence $\{b_p\}$ does not satisfy condition (H), i.e., that the series

$$\Sigma \mid b_{2p+1} \mid, \qquad \Sigma \mid b_{2p+1}(b_2 + b_4 + \cdots + b_{2p})^2 \mid \tag{2.1}$$

* Received August 20, 1946; Revised April 26, 1947.

converge, and that the sequence $s_p = b_2 + b_4 + \cdots + b_{2p}$, $p = 1, 2, 3, \cdots$, has at least one finite limit-point. Then the series $\Sigma \mid b_{2p+1}s_p \mid$ converges, and there exists an index n such that $\mid b_{2p+1}s_p \mid < 1$ for $p > n$, so that the infinite product $\prod_{p=1}^{\infty} (1 + b_{2n+2p+1}s_{n+p})$ converges and is not zero.

Let A_p and B_p denote the p-th numerator and denominator of the continued fraction. Then

$$A_{p+1} = b_{p+1}A_p + A_{p-1}, \quad B_{p+1} = b_{p+1}B_p + B_{p-1}, \qquad p = 0, 1, 2, \cdots, \tag{2.2}$$

where $A_{-1} = 1$, $B_{-1} = 0$, $A_0 = 0$, $B_0 = 1$. Put

$$\begin{aligned} &\pi_0 = 1, \quad \pi_p = \prod_{r=1}^{p} (1 + b_{2n+2r+1}s_{n+r}), && p = 1, 2, 3, \cdots, \\ &U_{2p} = A_{2n+2p+1}/\pi_p, && V_{2p} = B_{2n+2p+1}/\pi_p, \\ &U_{2p+1} = (A_{2n+2p+2} - s_{n+p+1}A_{2n+2p+1})\pi_p, \\ &V_{2p+1} = (B_{2n+2p+2} - s_{n+p+1}B_{2n+2p+1})\pi_p, \\ &c_{2p} = b_{2n+2p+1}/\pi_{p-1}\pi_p, && c_{2p+1} = -b_{2n+2p+1}s^2{}_{n+p}\pi_{p-1}\pi_p. \end{aligned} \tag{2.3}$$

One may then readily verify that

$$U_p = c_pU_{p-1} + U_{p-2}, \qquad V_p = c_pV_{p-1} + V_{p-2}, \qquad p = 2, 3, 4, \cdots. \tag{2.4}$$

Inasmuch as the series (2.1) are convergent and $\lim_{p=\infty} \pi_p = \rho$ exists and is finite and not zero, it follows that the series $\Sigma \mid c_p \mid$ is convergent. Hence we conclude by a known argument (cf., for instance, Hellinger and Wall [3], p. 113) that the sequences $\{U_{2p}\}$, $\{V_{2p}\}$, $\{U_{2p+1}\}$, $\{V_{2p+1}\}$ have finite limits X_0, Y_0, X_1, Y_1, respectively. Therefore, by (2.3),

$$\begin{aligned} &\lim_{p=\infty} A_{2p+1} = \rho X_0, && \lim_{p=\infty} (A_{2p} - s_pA_{2p-1}) = X_1/\rho, \\ &\lim_{p=\infty} B_{2p+1} = \rho Y_0, && \lim_{p=\infty} (B_{2p} - s_pB_{2p-1}) = Y_1/\rho. \end{aligned} \tag{2.5}$$

Inasmuch as $A_{2p-1}(B_{2p} - s_pB_{2p-1}) - B_{2p-1}(A_{2p} - s_pA_{2p-1}) = 1$, it follows that

$$X_0Y_1 - X_1Y_0 = 1. \tag{2.6}$$

Let s be a finite limit-point of the sequence $\{s_p\}$, and suppose that $s_p \to s$ as $p \to \infty$ over a sequence P of indices. We then conclude from (2.5) that

$$\begin{aligned} &\lim A_{2p} = s\rho X_0 + X_1/\rho = F(s), \\ &\lim B_{2p} = s\rho Y_0 + Y_1/\rho = G(s), \end{aligned} \tag{2.7}$$

as $p \to \infty$ over P. Moreover, by (2.6),

$$X_0 G(s) - Y_0 F(s) = 1,$$

and

$$F(s)G(t) - F(t)G(s) = s - t.$$

If condition (H) is not satisfied by the sequence $\{b_p\}$, we now conclude immediately that the sequence of approximants of the continued fraction has distinct limit-points X_0/Y_0, $F(s)/G(s)$, where the range of s is the set of finite limit-points of the sequence $\{s_p\}$. Hence condition (H) is necessary for the convergence of the continued fraction.

On applying the preceding considerations to the continued fraction $\overset{\infty}{\underset{p=2}{K}}(1/b_p)$, we conclude that $\overset{\infty}{\underset{p=1}{K}}(1/b_p)$ diverges if the series $\Sigma \mid b_{2p+2} \mid$ and $\Sigma \mid b_{2p+2}(b_3 + b_5 + \cdots + b_{2p+1})^2 \mid$ converge and the sequence $b_3 + b_5 + \cdots + b_{2p+1}$, $p = 1, 2, 3, \cdots$, has a finite limit-point. In particular, we have the following extension of a theorem of von Koch [4].

THEOREM B. *The continued fraction* $\overset{\infty}{\underset{p=1}{K}}(1/b_p)$ *diverges if the series* Σb_{2p} *and* Σb_{2p+1} *converge, at least one of them absolutely. When the condition is satisfied, the sequences* $\{A_{2p}\}$, $\{B_{2p}\}$, $\{A_{2p+1}\}$, $\{B_{2p+1}\}$ *of even and odd numerators and denominators have finite limits* F_0, G_0, F_1, G_1, *respectively, where* $F_1 G_0 - F_0 G_1 = 1$.

3. The question of sufficiency of condition (H). The sequence $b_p = i$, $p = 1, 2, 3, \cdots$, satisfies condition (H) although the continued fraction $\overset{\infty}{\underset{p=1}{K}}(1/b_p)$ diverges in this case. Hence condition (H) is not, in general, sufficient for the convergence of the continued fraction. This is shown also by the fact that the continued fraction diverges if all its *odd* partial denominators b_{2p-1} are zero. In this case, the series (2.1) are both convergent, and condition (H) therefore reduces to the condition

$$\lim_{p=\infty} \mid b_2 + b_4 + \cdots + b_{2p} \mid = \infty. \tag{3.1}$$

The following theorem covers a class of *convergent* continued fractions for which condition (H) takes this form.

THEOREM C. *If* $R(b_1) > 0$, $R(b_p) \geq 0$, $p = 2, 3, 4, \cdots$, *and if the series* (2.1) *are both convergent, then the continued fraction* $\overset{\infty}{\underset{p=1}{K}}(1/b_p)$ *converges if and only if* (3.1) *holds.*

Proof. The fact that (3.1) is necessary for convergence of the continued fraction is contained in Theorem A. Moreover, from (2.5) we see that when (3.1) holds, then

$$\lim_{p=\infty} \frac{A_p}{B_p} = \frac{X_0}{Y_0}.$$

The finite numbers X_0 and Y_0 are not both zero, by virtue of (2.6). Since $R(b_p) \geq 0$, the linear transformation $t = t_p(w) = 1/(b_p + w)$ of the w-plane into the t-plane maps $R(w) \geq 0$ into $R(t) \geq 0$; since $R(b_1) > 0$, $t = t_1(w)$ maps $R(w) \geq 0$ upon the circular region

$$\left| t - \frac{1}{2R(b_1)} \right| \leq \frac{1}{2R(b_1)}.$$

It follows that the transformation $t = t_1 t_2 \cdots t_p(w)$ maps $R(w) \geq 0$ into this circular region. Therefore, since $t_1 t_2 \cdots t_p(0) = A_p/B_p$, we conclude that

$$\left| \frac{A_p}{B_p} \right| \leq \frac{1}{R(b_1)}, \quad p = 1, 2, 3, \cdots. \tag{3.2}$$

Consequently, $Y_0 \neq 0$, so that the continued fraction converges to the value X_0/Y_0.

Remark. It is easy to see that the conclusion in Theorem C holds if $b_1 = b_3 = \cdots = b_{2m-1} = 0$, $R(b_{2m+1}) > 0$, $R(b_p) \geq 0$ for $p > 2m + 1$, the series (2.1) are both convergent, and $\lim s_p = \infty$.

If at least one of the series (2.1) is divergent, we can establish convergence of the continued fraction $\overset{\infty}{\underset{p=1}{K}} (1/b_p)$ under somewhat more restrictive hypotheses upon the b_p than were used in Theorem C.

THEOREM D. *Let* $b_{2p-1} = k_{2p-1} z_p$, $b_{2p} = k_{2p}$, $p = 1, 2, 3, \cdots$, *where*

$$k_1 > 0,\ k_{2p+1} \geq 0,\ R(k_{2p}) \geq 0,\ R(z_p) \geq \delta,\ |z_p| \leq M,\ p = 1, 2, 3, \cdots, \tag{3.3}$$

δ *and* M *being positive constants. The continued fraction* $\overset{\infty}{\underset{p=1}{K}} (1/b_p)$ *converges if and only if the sequence* $\{k_p\}$ *satisfies condition* (H).

Remark. The continued fraction $\overset{\infty}{\underset{p=1}{K}} (1/b_p)$ is equivalent to the continued fraction $i \cdot \overset{\infty}{\underset{p=1}{K}} (1/b'_p)$, where $b'_{2p-1} = ib_{2p-1}$, $b'_{2p} = -ib_{2p}$, $p = 1, 2, 3, \cdots$. The sequence $\{b'_p\}$ evidently satisfies condition (H) if and only if the

sequence $\{b_p\}$ satisfies condition (H). We may therefore replace the conditions (3.3) by the conditions

(3.4) $k_1 > 0,\ k_{2p+1} \geq 0,\ I(k_{2p}) \leq 0,\ I(z_p) \geq \delta,\ |z_p| \leq M,\ p = 1, 2, 3, \cdots.$

Proof of Theorem D. By Theorem A, it is necessary for the convergence of the continued fraction that the sequence $\{k_p\}$ satisfy condition (H). In case the series

(3.5) $$\Sigma k_{2p+1}, \qquad \Sigma k_{2p+1} \,|\, k_2 + k_4 + \cdots + k_{2p} \,|^2$$

converge and

(3.6) $$\lim_{p=\infty} |\, k_2 + k_4 + \cdots + k_{2p} \,| = \infty,$$

then the continued fraction converges by Theorem C. Hence there is to be considered here the case where at least one of the series (3.5) is divergent.

The inequality (3.2) is now

(3.7) $$\left| \frac{A_p}{B_p} \right| \leq \frac{1}{k_1 \delta}, \quad p = 1, 2, 3, \cdots,$$

so that $B_p \neq 0,\ p = 1, 2, 3, \cdots.$

We have to consider two cases (A) and (B) according as $k_p \neq 0$ for all values of p, or $k_p = 0$ for one or more values of p.

(A) *Suppose that* $k_p \neq 0$ *for* $p = 1, 2, 3, \cdots.$ Put

(3.8) $$c_0 = 0, \qquad c_p = 1/k_p k_{p+1}, \qquad p = 1, 2, 3, \cdots,$$

(3.9) $$T_0(w) = 1/k_1 w, \qquad T_1(w) = c_1 + z_1 - \frac{c_1 c_2}{w},$$

$$T_p(w) = c_{2p-1} + c_{2p-2} + z_p - \frac{c_{2p} c_{2p-1}}{w}, \qquad p = 2, 3, 4, \cdots.$$

One may then verify that the approximants of $\overset{\infty}{\underset{p=1}{K}}(1/b_p)$ are given by the formulas

(3.10) $$\begin{aligned} T_0 T_1 \cdots T_p(z_{p+1} + c_{2p} + c_{2p+1}) &= A_{2p+2}/B_{2p+2}, \\ T_0 T_1 \cdots T_p(z_{p+1} + c_{2p}) &= A_{2p+1}/B_{2p+1}, \end{aligned} \qquad p = 0, 1, 2, \cdots.$$

Under the hypothesis that (3.4) holds, we have

$$I(c_p) = \gamma_p \geq 0, \qquad p = 0, 1, 2, \cdots.$$

Let

$$\beta_p = \gamma_{2p-2} + \gamma_{2p-1},$$

$$g_0 = 0,\ g_p = \begin{cases} 1 \text{ if } \gamma_{2p+1} = 0, \\ \gamma_{2p}/(\gamma_{2p} + \gamma_{2p+1}) \text{ if } \gamma_{2p+1} > 0, \end{cases} \qquad p = 1, 2, 3, \cdots.$$

$$\alpha_p = I[(c_{2p-1} c_{2p})^{1/2}],$$

Then we find that $0 \leq g_{p-1} \leq 1$ and $\alpha^2_p = \gamma_{2p-1}\gamma_{2p} = \beta_p\beta_{p+1}(1 - g_{p-1})g_p$, $p = 1, 2, 3, \cdots$. Thus the continued fraction generated by the sequence of transformations (3.9) is a *positive definite* continued fraction, (cf. [1] p. 265). Accordingly, we may construct a nest of circular regions $K_0 \supset K_1 \supset K_2 \supset K_3 \supset \cdots$, where K_p is the image of the half-plane $I(w) \geq \beta_{p+1}g_p = \gamma_{2p} = I(1/k_{2p}k_{2p+1})$ under the transformation $t = T_0T_1 \cdots T_p(w)$. Inasmuch as $I(z_{p+1} + c_{2p} + c_{2p+1}) > \gamma_{2p}$, $I(z_{p+1} + c_{2p}) > \gamma_{2p}$, we conclude by (3.10) that A_{2p+2}/B_{2p+2} and A_{2p+1}/B_{2p+1} have their values in K_p. To prove that the continued fraction $\overset{\infty}{\underset{p=1}{K}}(1/b_p)$ converges, it therefore suffices to prove that the radius r_p of K_p has the limit zero for $p = \infty$.

Let $u_0 = 1$, $u_p = (c_{2p-1}c_{2p})^{\frac{1}{2}}$, $v_p = c_{2p-2} + c_{2p-1}$, $p = 1, 2, 3, \cdots$; $Y_0(z) = 0$, $Y_1(z) = 1$,

$$-u_{p-1}Y_{p-1}(z) + (v_p + z_p)Y_p(z) - u_pY_{p+1}(z) = 0, \qquad p = 1, 2, 3, \cdots.$$

Then the radius r_p of K_p satisfies the inequality

$$r_p \leq \frac{1}{2\sum_{r=1}^{p} I(z_r)\,|\,Y_r(z)\,|^2} \leq \frac{1}{2\delta\sum_{r=1}^{p} |\,Y_r(z)\,|^2}.$$

This can be established by means of an easy extension of formula (3.12) of [1]. Hence $\lim_{p=\infty} r_p = 0$ provided the infinite series $\sum_{r=1}^{\infty} |\,Y_r(z)\,|^2$ is divergent. Let $X_0(z) = -1$, $X_1(z) = 0$,

$$-u_{p-1}X_{p-1}(z) + (v_p + z_p)X_p(z) - u_pY_{p+1}(z) = 0, \qquad p = 1, 2, 3, \cdots.$$

Then, as in formula (3.4) of [1], we have

$$\left|\frac{X_p(z)}{Y_p(z)}\right| \leq \frac{1}{I(z_1)} \leq \frac{1}{\delta}.$$

It follows that the series $\sum_{r=1}^{\infty} |\,Y_r(z)\,|^2$ diverges if the series $\sum_{r=1}^{\infty} |\,X_r(z)|^2$ diverges. Thus $\lim_{p=\infty} r_p = 0$, provided at least one of the series

$$(3.11) \qquad \sum_{r=1}^{\infty} |\,X_r(z)\,|^2, \qquad \sum_{r=1}^{\infty} |\,Y_r(z)\,|^2$$

is divergent. Now one may prove by mathematical induction that when $z_p = 0$, $p = 1, 2, 3, \cdots$, the series (3.11) are precisely the series (3.5). Hence the proof of Theorem D in the case under consideration may be immediately completed by means of the following *theorem of invariability*.

THEOREM 3.1. *Let $u_0 = 1, u_1, u_2, u_3, \cdots$ be constants different from zero, and let $v_1, v_2, v_3, \cdots$ be constants. Let $x_p = X_p(z)$, $x_p = Y_p(z)$ be solutions of the equations $-u_{p-1}x_{p-1} + (v_p + z_p)x_p - u_p x_{p+1} = 0$, $p = 1, 2, 3, \cdots$ such that $X_0(z) = -1$, $X_1(z) = 0$, $Y_0(z) = 0$, $Y_1(z) = 1$. If the two series $\Sigma |X_p(z)|^2$ and $\Sigma |Y_p(z)|^2$ converge for $z_p = h_p$, $p = 1, 2, 3, \cdots$, then these series converge uniformly for $|z_p - h_p| < M$, $p = 1, 2, 3, \cdots$, where M is any positive constant.*

This is a generalization of a theorem of Hellinger and Wall [3], and may be established by the method used in [3].

(B) *Suppose that $k_p = 0$ for one or more values of p.* Let

$$t_{2p}(w) = 1/(k_{2p} + w), \quad t_{2p-1}(w) = 1/(k_{2p-1}z_p + w), \qquad p = 1, 2, 3, \cdots.$$

The n-th approximant of the continued fraction is given by $A_n/B_n = t_1 t_2 \cdots t_n(0)$, $n = 1, 2, 3, \cdots$.

We suppose first that there exists an index m such that

$$k_{2m-1} > 0, \; k_{2m} + k_{2m+2} + \cdots + k_{2m+2n} \neq 0 \text{ implies that } k_{2m+2n+1} = 0, \quad n = 0, 1, 2, \cdots.$$

Then, for all n sufficiently large,

$$\frac{A_n}{B_n} = t_1 t_2 \cdots t_{2m-2} \left(\frac{1}{k_{2m-1}z_m + k_{2m+1}z_{m+1} + \cdots + k_{2m+2r+1}z_{m+r+1} + w_n} \right),$$

where, if (3.4) holds, $I(w_n) \geq 0$, and where $r \to \infty$ as $n \to \infty$. Since

$$\sum_{p=1}^{n} k_{2p+1} |k_2 + k_4 + \cdots + k_{2p}|^2 = \sum_{p=1}^{m-1} k_{2p+1} |k_2 + k_4 + \cdots + k_{2p}|^2 + |k_2 + k_4 + \cdots + k_{2m-2}|^2 \sum_{p=m}^{n} k_{2p+1},$$

it follows that the first of the series (3.5) is divergent. We therefore conclude immediately that

$$\lim_{n=\infty} \frac{A_n}{B_n} = \frac{A_{2m-2}}{B_{2m-2}},$$

which is finite by (3.7). The continued fraction is thus convergent in case such an idex m exists.

If there is no index m with the specified property, then there is a continued fraction

(3.12) $$\underset{p=1}{\overset{\infty}{K}} (1/b'_p), \qquad b'_{2p-1} = k'_{2p-1}z'_p, \qquad b'_{2p} = k'_{2p},$$

where k'_{2p-1}, k'_{2p}, z'_p are of the form

$$(3.13) \qquad k'_{2p-1} = \sum_{r=\alpha}^{r=\beta} k_{2r+1} > 0,\ k'_{2p} = \sum_{r=\beta}^{r=\gamma} k_{2r+2} \neq 0,\ z'_p = \frac{\sum_{r=\alpha}^{r=\beta} k_{2r+1} z_{r+1}}{k'_{2p-1}},$$

whose sequence of approximants, $\{A'_n/B'_n\}$, is a subsequence of the sequence $\{A_n/B_n\}$. By (3.4) and (3.13), $I(k'_{2p}) \leq 0$, $I(z'_p) \geq \delta$. Also, at least one of the series $\Sigma k'_{2p+1}$ or $\Sigma k'_{2p+1} \mid k'_2 + k'_4 + \cdots + k'_{2p} \mid^2$ is divergent. Hence, by (A), the continued fraction (3.12) converges to a value L, the radii r'_p of the circles K'_p associated with (3.12) have the limit 0 for $p = \infty$, and their centers have the limit L for $p = \infty$. Those approximants in the sequence $\{A_n/B_n\}$ falling between A'_{2p}/B'_{2p} and A'_{2p+2}/B'_{2p+2} are all of the form $t'_1 t'_2 \cdots t'_{2p}(1/u)$, $t'_n(w) = 1/(b'_n + w)$, where $I(u) \geqq 0$. Since $t'_1 t'_2 \cdots t'_{2p}(1/u)$ is the $2p$-th approximant of (3.12) in which b'_{2p} has been replaced by $b'_{2p} + (1/u)$, it is given by the expression

$$(3.14) \qquad T'_{p-1}(s), \qquad s = z'_p + \frac{1}{k'_{2p-2} k'_{2p-1}} + \frac{1}{k'_{2p-1}(k'_{2p} + 1/u)},$$

where T'_{p-1} is the transformation defined in (3.9), formed for the continued fraction (3.12). Inasmuch as $I(s) \geq I(1/k'_{2p-2} k'_{2p-1})$, we conclude immediately that the value of (3.14) is in the circle K'_{p-1}. It therefore follows that $\lim_{n=\infty} (A_n/B_n) = L$, so that the continued fraction is convergent.

This completes the proof of Theorem D.

Remark. The condition "$k_1 > 0$, $k_{2p+1} \geq 0$, $p = 1, 2, 3, \cdots$" in Theorem D may be replaced by "$k_{2p-1} \geq 0$, $p = 1, 2, 3, \cdots$, with inequality for at least one p." If, however, $k_{2p-1} = 0$, $p = 1, 2, 3, \cdots$, then the continued fraction is divergent.

As in the proof of Theorem C, we see that under the hypothesis of Theorem D, all the approximants of the continued fraction have their values in the circle

$$(3.15) \qquad \left| w - \frac{1}{2k_1\delta} \right| = \frac{1}{2k_1\delta}.$$

Therefore, by Theorem D and a well known theorem on uniformly bounded sequences of analytic functions, we have the following theorem.

THEOREM E. *If $k_1 > 0$, $k_{2p+1} \geq 0$, $R(k_{2p}) \geq 0$, $p = 1, 2, 3, \cdots$, and the sequence $\{k_p\}$ satisfies condition (H), then the continued fraction* $\overset{\infty}{K}_{p=1} (1/b_p)$, $b_{2p-1} = k_{2p-1} z$, $b_{2p} = k_{2p}$, *converges uniformly over every bounded closed region in the right half-plane $R(z) > 0$, and its value is an analytic*

function of z in this half-plane, which, for $R(z) \geq \delta > 0$, has its values in the circle (3.15).

4. Older theorems derived from Theorem D. We shall now obtain four older convergence theorems as easy consequences of Theorem D.

THEOREM 4.1. (Stieltjes [7]) *Let us suppose that $k_1 > 0$, $k_p \geq 0$, $p = 1, 2, 3, \cdots$. If the series Σk_p diverges, then the continued fraction $\overset{\infty}{\underset{p=1}{K}}(1/b_p)$, $b_{2p-1} = k_{2p-1}z$, $b_{2p} = k_{2p}$, converges uniformly over every bounded closed region G of the z-plane whose distance from the negative half of the real axis is positive. If the series Σk_p converges, then the continued fraction diverges for every value of z.*

Proof. The sequence $\{k_p\}$ satisfies condition (H) if and only if the series Σk_p diverges, and therefore, by Theorem A, the divergence of this series is necessary for the convergence of the continued fraction. Let the distance of G from the negative half of the real axis be δ, and let $G = G_1 + G_2 + G_3$, where G_1 is in $R(z) \geq \delta$, G_2 is in $I(z) \geq \delta$, and G_3 is in $I(z) \leq -\delta$. If $\{k_p\}$ satisfies condition (H), the continued fraction converges uniformly over G_1 by Theorem E. Since

$$\overset{\infty}{\underset{p=1}{K}}(1/b_p) = \begin{cases} -i \cdot \overset{\infty}{\underset{p=1}{K}}(1/b'_p),\ b'_{2p-1} = k_{2p-1}(-iz),\ b'_{2p} = ik_{2p}, \\ i \cdot \overset{\infty}{\underset{p=1}{K}}(1/b''_p),\ b''_{2p-1} = k_{2p-1}(iz),\ b''_{2p} = -ik_{2p}, \end{cases}$$

the same conclusion holds for G_2 and G_3, and therefore for G.

THEOREM 4.2. (Van Vleck [10]) *If $R(b_1) > 0$, $|I(b_p)| \leq cR(b_p)$, $p = 1, 2, 3, \cdots$, where c is a positive constant, then the continued fraction $\overset{\infty}{\underset{p=1}{K}}(1/b_p)$ converges if and only if the infinite series $\Sigma |b_p|$ is divergent.*

Proof. In Theorem D, take $k_{2p} = b_{2p}$, $k_{2p-1} = |b_{2p-1}|$, $k_{2p-1}z_p = b_{2p-1}$, where $z_p = 1$ if $b_{2p-1} = 0$. Then $R(k_{2p}) \geq |I(b_{2p})|/c \geq 0$, $k_1 = |b_1| > 0$, $k_{2p+1} = |b_{2p+1}| \geq 0$, $|z_p| = 1 < M$, $R(z_p) \geq 1/(1 + c^2)^{\frac{1}{2}} = \delta$. If the series $\Sigma |b_p|$ diverges, then either Σk_{2p+1} diverges, or else $\Sigma |b_{2p}|$ diverges, in which case $\lim |k_2 + k_4 + \cdots + k_{2p}| = \infty$ inasmuch as $|b_{2p}| \leq (1 + c^2)^{\frac{1}{2}} R(b_{2p})$. We then conclude from Theorem D that the continued fraction converges if the series $\Sigma |b_p|$ diverges. If this series converges, then the continued fraction diverges by Theorem A.

Remark. Theorem 4.1 can be derived from Theorem 4.2, as Van Vleck showed.

If in Theorem 4.2 we require that $|I(b_p)| \leq cR(b_p)$ only for *odd* values of p, we obtain the following theorem.

THEOREM 4.3. (Mall [5]) *If* $R(b_1) > 0$, $|I(b_{2p-1})| \leq cR(b_{2p-1})$, $R(b_{2p}) \geq 0$, $p = 1, 2, 3, \cdots$, *where* c *is a positive constant, then the continued fraction* $\overset{\infty}{\underset{p=1}{K}} (1/b_p)$ *converges if and only if the sequence* $\{b_p\}$ *satisfies condition* (H).

Proof. Choose the k_p and z_p as in the proof of Theorem 4.2, and apply Theorem D.

THEOREM 4.4. (Hamburger [2]) *If the* k_p *are real,* $k_1 > 0$, $k_{2p+1} \geq 0$, $p = 1, 2, 3, \cdots$, *then the continued fraction of Theorem* 4.1 *converges for all nonreal* z *if and only if the sequence* $\{k_p\}$ *satisfies condition* (H). *When the condition is satisfied, the continued fraction converges uniformly over every closed bounded region whose distance from the real axis is positive.*

Proof. Use the transformations employed in the proof of Theorem 4.1, and apply Theorem E.

Remark. Theorems 4.1, 4.3 and 4.4 are somewhat more general than are to be found in the papers of Stieltjes, Mall and Hamburger, respectively.

The results found in this paper serve again to emphasize the importance of the theory of positive definite continued fractions. This theory overlaps or contains the greater part of the existing analytic theory of continued fractions. In this connection, we mention the following theorem of Thron [8].

THEOREM 4.5. *Let* a *and* b *be real numbers of which* $a > 0$, *and suppose that* $I(b_p) \geq b$, $|I(a_p)| \leq a/2$, $p = 1, 2, 3, \cdots$. *If there exists a constant* M *such that* $|a_p| < M$, $p = 1, 2, 3, \cdots$, *then the J-fraction* $\overset{\infty}{\underset{p=1}{K}} (-a^2_{p-1}/(b_p + z))$, $a^2_0 = -1$, *converges uniformly over every bounded closed region in the half-plane* $I(z) > a - b$.

As indicated (without proof) by Thron ([8], p. 413; cf. also [9], p. 779), this theorem is contained in the general theory of positive definite continued fractions. It is only necessary to put $z = (a - b)i + z'$, $\beta_p = I(b_p) + a - b$, $\alpha_p = I(a_p)$, and we see that $\beta_p \geq 0$, $\alpha^2_p \leq \beta_p \beta_{p+1} \cdot \frac{1}{2}(1 - \frac{1}{2})$, $p = 1, 2, 3, \cdots$,

so that the J-fraction of Thron is a particular positive definite J-fraction in the variable z'. It therefore converges for $I(z') > 0$, i. e. for $I(z) > a - b$, inasmuch as the series $\Sigma(1/|a_p|)$ diverges when $|a_p| < M$, $p = 1, 2, 3, \cdots$.

NORTHWESTERN UNIVERSITY
AND
THE UNIVERSITY OF TEXAS.

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LEFT ASSOCIATED MATRICES WITH ELEMENTS IN AN ALGEBRAIC DOMAIN.*

By B. M. STEWART.

I. A Necessary and Sufficient Condition.

1. Introduction. For the set M of all n-by-n matrices, with elements in an algebraic field F of order k over the rational field Ra, all the usual definitions and properties of matrices with elements in an infinite field will hold. But for that subset $[M]$ of matrices whose elements are in the corresponding algebraic domain $[F]$ some of these properties have a special character. For example, every *non-singular* matrix P of M with determinant $d(P) \neq 0$ has an inverse matrix P^{-1} in M such that $PP^{-1} = P^{-1}P = I$, where I is the identity matrix of M; but a matrix P of $[M]$, if it has an inverse matrix P^{-1} in $[M]$, must be not only non-singular, but also *unimodular*, i.e., the determinant $d(P)$ must be a unit of $[F]$.

As a further example, there is in the set M the notion of *rank*, but in the set $[M]$, the special notion of an *irreducible basis*. A matrix A of M is said to be of rank r if $n - r$ is the maximum number of zero rows to be found in any matrix PA where P is non-singular in M; a matrix A of $[M]$ is said to have an irreducible basis of b rows if $n - b$ is the maximum number of zero rows to be found in any matrix PA where P is unimodular in $[M]$. As will be described in **7**, if the domain $[F]$ is a principal ideal ring, then b is always equal to r; but if $[F]$ is not a principal ideal ring, then there are matrices for which $b = r + 1$. This anomaly is of importance in the present problem.

Two matrices A and B of $[M]$ are said to be left associates in $[M]$ if there exists a unimodular matrix P of $[M]$ such that $PA = B$. This notion is an equals relation dividing the matrices of $[M]$ into mutually exclusive classes of left associated matrices. Our fundamental problem is to determine necessary and sufficient conditions for this class division.

If the domain under consideration is a principal ideal ring, a necessary and sufficient condition that A and B be left associates is that A and B have

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the same Hermite triangular canonical form (zeros above the main diagonal; if a diagonal element is zero, then its row is zero; elements below the diagonal in a prescribed set of residues modulo the diagonal element above).[1] However, this test is a practical one only if there is an algorithm for determining a greatest common divisor, such as the Euclidean algorithm for the rational domain $[Ra]$; and for domains whose class number is greater than one, the presence of non-principal ideals prevents the *direct* solution of the problem by an Hermite form.

However, if each element of a matrix A of $[M]$ is replaced by its k-by-k second matric representation, there is produced an *enlarged* matrix A^E, of order kn-by-kn, to be sure, but with elements in the rational domain! Hence for A^E the Hermite form is well-defined *and* easily found. In this paper we prove that a necessary and sufficient condition that matrices A and B of $[M]$ be left associates in $[M]$ is that the corresponding enlarged matrices A^E and B^E be left associates, i. e., that the enlarged matrices have the same Hermite form. Although the proof and the actual construction of a unimodular transforming matrix P (whose construction we defer to Part II) depend upon other considerations, the test itself involves only the rational domain and affords a practical criterion for the division of the matrices of $[M]$ into classes of left associated matrices.

The proof is accomplished by considering the more primitive concept of *mutual left. divisibility*: two matrices A and B of $[M]$ are called mutually left divisible if there exist matrices P and Q of $[M]$ such that $PA = B$ and $QB = A$. We adapt from the literature the proof that mutual left divisibility of A and B in $[M]$ is equivalent to left associativity of A and B in $[M]$. We prove that mutual left divisibility of A and B in $[M]$ is equivalent to the left associativity of A^E and B^E. Hence our principal result follows.

Our problem is related to the papers of Steinitz,[2] Krull,[3] and Franz[4] who have solved the problems of finding for matrices of $[M]$ the necessary and sufficient conditions for *divisibility*: $PAQ = B$; for *mutual divisibility*: $PAQ = B$ and $SBT = A$; and for *equivalence*: $PAQ = B$, with P and Q

[1] C. C. MacDuffee, "Matrices with elements in a principal ideal ring," *Bulletin of the American Mathematical Society*, vol. 39 (1933), pp. 564-584.

[2] E. Steinitz, Recteckige Systeme und Moduln in algebraischen Zahlkörpern," *Mathematische Annalen*, vol. 71 (1911), pp. 328-354, and vol. 72 (1912), pp. 297-345.

[3] W. Krull, "Matrizen, Moduln und verallgemeinerte Abelsche Gruppen in Bereich der ganzen algebraischen Zahlen," *Sitzungberichte Heidelberg Akademie der Wissenschaften*, 2 Abhandlung (1932), pp. 13-18.

[4] W. Franz, "Elementarteilertheorie in algebraischen Zahlkörpern," *Journal für die reine und angewandte Mathematik*, vol. 171 (1934), pp. 149-161.

both unimodular. Obviously left associativity is a special type of equivalence. With these authors, we have no hesitation in using a canonical form enlarged beyond that used when the domain is a principal ideal ring. The new feature is that our canonical form has elements not in the original domain $[F]$, but in the rational domain $[Ra]$.

2. Second matric representation. If the algebraic field F is of order k, then a set of k elements, $e_1 = 1, e_2, \cdots, e_k$, in the domain $[F]$, can be found to serve as a *basis* for the field in the sense that for every element a of F there is a unique set of k numbers a_i in the rational field Ra such that

$$a = a_1e_1 + a_2e_2 + \cdots + a_ke_k; \tag{1}$$

further, a is in $[F]$, if and only if each a_i of (1) is in the rational domain $[Ra]$.[5]

Each product e_ie_j is in $[F]$, and hence can be written in the form

$$e_ie_j = \sum_{m=1}^{k} c_{ijm}e_m, \qquad (i, j = 1, \cdots, k), \tag{2}$$

where the k^3 *constants of multiplication* c_{ijm} are in $[Ra]$.

If we define S_i to be the k-by-k matrix $S_i = (c_{ris})$, then the correspondence

$$a \leftrightarrow S(a) = a_1S_1 + a_2S_2 + \cdots + a_kS_k,$$

where the a_i are the same as in (1), defines the so-called *second matric representation* of F, for it is well-known that the system of k-by-k matrices $S(a)$ with matric addition and multiplication as operations is isomorphic with the field F.

Since $e_1 = 1$, the first row of S_i has $c_{1is} = d_{is}$ where $d_{is} = 0$ if $i \neq s$ and $d_{ii} = 1$. Thus the first row of $S(a)$ is exactly $(a_1, a_2, \cdots, a_k)$. In other words, the matrix $S(a)$ is completely determined by its first row. In particular, $S(a)$ represents a number of $[F]$ if and only if all the elements of $S(a)$ are in $[Ra]$. These facts provide the basis for our fundamental lemma.

LEMMA. *If P is a k-by-kn matrix with elements in $[Ra]$ such that*

[5] For example, see E. Landau, *Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale*, Part I, Teubner (1918).

$$P \begin{pmatrix} S(a_1) \\ \cdot \\ \cdot \\ \cdot \\ S(a_n) \end{pmatrix} = S(b),$$

where $a_1, a_2, \cdots, a_n, b$ *are in* $[F]$, *then there exist* n *numbers* $p_1, p_2, \cdots, p_n$ *in* $[F]$ *such that*

$$(S(p_1) \cdots S(p_n)) \begin{pmatrix} S(a_1) \\ \cdot \\ \cdot \\ \cdot \\ S(a_n) \end{pmatrix} = S(b).$$

Separate the *first* row of P into n blocks each 1-by-k, say $p_{11}, \cdots, p_{1k}$; $p_{21}, \cdots, p_{2k}; \cdots; p_{n1}, \cdots, p_{nk}$; and define $p_i = p_{i1}e_1 + \cdots + p_{ik}e_k$, $i = 1, 2, \cdots, n$. Evidently p_i is in $[F]$, for each p_{ij} is in $[Ra]$. The first row of the matrix $(S(p_1) \cdots S(p_n))$ is the same as the first row of the matrix P; hence, by the hypothesis concerning the given matric product, the new matric product agrees in its first row with the first row of $S(b)$. But the new product considered in terms of k-by-k blocks is the sum and product of S-matrices and is therefore, by the isomorphism explained above, an S-matrix. Finally, since an S-matrix is completely determined by its first row, the new product must be exactly $S(b)$.

3. Enlarged matrices. If each element of a matrix A of M is replaced by its second matric representation there is produced an *enlarged matrix* A^E which is kn-by-kn with elements in Ra. Conversely, to a matrix A^E which is kn-by-kn with elements in Ra and each of whose k-by-k blocks is in S-form there corresponds an n-by-n matrix A of M.

If A, B, C are matrices of M such that $AB = C$, then $A^E B^E = C^E$; and conversely. This follows from the possibility of block-multiplication of matrices.

If A is non-singular in M, then A^E is non-singular. For $AA^{-1} = I$ implies $A^E(A^{-1})^E = I^E$, hence $(A^E)^{-1} = (A^{-1})^E$.

If A is of rank r in M, then A^E is of rank kr. For there exist non-singular matrices P and Q of M such that

$$PAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}. \quad \text{Then} \quad P^E A^E Q^E = \begin{pmatrix} I_r{}^E & 0^E \\ 0^E & 0^E \end{pmatrix}.$$

But P^E and Q^E are non-singular, hence rank-preserving. Therefore the rank of A^E is the same as the rank of $I_r{}^E$.

If A is in $[M]$, then all the elements of A^E are in $[Ra]$; and conversely. For if $A = (a_{rs})$, then all the elements of $S(a_{rs})$ are in $[Ra]$ if and only if a_{rs} is in $[F]$.

If P is unimodular in $[M]$, then P^E is unimodular in $[Ra]$; and conversely. For if P^{-1} is in $[M]$, then $(P^{-1})^E = (P^E)^{-1}$ has all elements in $[Ra]$; and conversely.

4. Steinitz: On mutual left divisibility. From the work of Steinitz we borrow the following theorem:

If A, B, U, V are matrices of $[M]$ such that $UA = B$ and $VB = A$, then there exists a unimodular matrix P in $[M]$ such that $PA = B$.

The converse is evident. Therefore, using the terms of **1**, we may state that the matrices A and B of $[M]$ are left associates in $[M]$ if and only if they are mutually left divisible in $[M]$.

5. Left associated enlarged matrices. We are now in a position to establish the following theorem:

A necessary and sufficient condition that the matrices A and B of $[M]$ be mutually left divisible in $[M]$ is that the enlarged matrices A^E and B^E be left associates in $[Ra]$.

If A and B are mutually left divisible in $[M]$, then there exist matrices U and V of $[M]$ such that $UA = B$ and $VB = A$. Then by **4**, there exists a unimodular matrix P of $[M]$ such that $PA = B$. Hence by **3** we have $P^E A^E = B^E$ with P^E unimodular in $[Ra]$. Therefore A^E and B^E are left associates in $[Ra]$.

Conversely, if A^E and B^E are left associates in $[Ra]$, then they have the same Hermite form H and there exist matrices X and Y, kn-by-kn and unimodular in $[Ra]$, such that $XA^E = H = YB^E$. Then $U' = Y^{-1}X$ and $V' = X^{-1}Y$ are unimodular in $[Ra]$ and such that $U'A^E = B^E$ and $V'B^E = A^E$. However, the matrices U' and V' do not, in general, have their k-by-k blocks in S-form. Therefore by repeated applications of the fundamental lemma in **2**, we construct new transforming matrices that will be in S-form. We separate the matrices U' and V' into k-by-kn row blocks. To each of these we apply the lemma, i. e., using only the *first* rows of these row blocks, we define the rows of two matrices of $[M]$ which we shall designate as U and V.

By the lemma, these new matrices are such that $U^E A^E = B^E$ and $V^E B^E = A^E$. Therefore U and V are such that $UA = B$ and $VB = A$. Hence A and B are mutually left divisible in $[M]$.

Although U' and V' are unimodular in $[Ra]$, it does not follow, in general, that the matrices U^E and V^E, derived from U' and V' in the manner of the lemma, enjoy the same property. Therefore we cannot at once conclude that A and B are left associates. But by combining this theorem with the theorem of **4**, we have proved the following:

THEOREM I. *A necessary and sufficient condition that the matrices A and B of $[M]$ be left associates in $[M]$ is that the enlarged matrices A^E and B^E have the same Hermite form.*

Having applied this test to a pair of matrices A and B of $[M]$ and having discovered that A^E and B^E have the same Hermite form H, so that the existence of a matrix P, unimodular in $[M]$ and such that $PA = B$, is known, we may demand an explicit construction of the transforming matrix P. Toward this end we must distinguish various cases according to the rank of A and B. The rank of A and B can be immediately determined from the rank of H, for as we have noted in **3** the rank of the enlarged matrices is k-fold that of the original matrices. More than this can be said, for in the process of finding H we work on the columns of A^E from right to left; if we pause after each set of k columns, we must find that either k or no independent columns have been added to H, as will be evidenced by a set of k non-zero or a set of k zero entries, respectively, in the principal diagonal of H. By reference to H we can determine the independent columns of A and B, considered from right to left and occurring in the same positions for both A and B. Then, in particular, we may suppose, for subsequent convenience of notation, that the corresponding columns of A and B are rearranged so that, if the common rank is r, there are r independent columns of A and B at the extreme right of the matrices. Matrices A and B, thus arranged, will be said to be in *preferred form*. A matrix A can be recognized to be in preferred form if the Hermite form H of A^E has all the non-zero elements of the principal diagonal in the kr places at the extreme right, so that

$$H = \begin{pmatrix} 0 & 0 \\ H_2 & H_1 \end{pmatrix} \tag{3}$$

where H_1 is kr-by-kr, triangular, and of rank kr.

II. Construction of the Transforming Matrix.

6. The non-singular case. If the matrices A and B of $[M]$ are non-singular, we can give a direct proof, independent of the Steinitz theory, that the matrices A and B are left associates in $[M]$ if and only if the enlarged matrices have the same Hermite form. At the same time we find an explicit construction within $[Ra]$ for the transforming unimodular matrix P of $[M]$ such that $PA = B$.

Suppose A^E and B^E have the same Hermite form H so that there exist matrices X and Y, kn-by-kn, unimodular in $[Ra]$, such that $XA^E = H = YB^E$. Then $P' = Y^{-1}X$ is unimodular in $[Ra]$ and $P'A^E = B^E$. Since A is non-singular in M, A^E is non-singular (see **3**) and $(A^E)^{-1} = (A^{-1})^E$. Therefore $P' = B^E(A^E)^{-1} = B^E(A^{-1})^E$, and we see that P' has its k-by-k blocks in S-form and hence can be written $P' = P^E$. Since P' has all its elements in $[Ra]$, P is in $[M]$. Furthermore, P' is unimodular in $[Ra]$, hence P is unimodular in $[M]$ (see **3**). From $P'A^E = P^EA^E = B^E$ it follows that $PA = B$. Therefore A and B are left associates in $[M]$.

Conversely, if A and B are left associates in $[M]$ so that $PA = B$ with P unimodular in $[M]$, then $P^EA^E = B^E$ with P^E unimodular in $[Ra]$, so A^E and B^E are left associates in $[Ra]$ and must have the same Hermite form.

THEOREM II. *If A and B are non-singular matrices of $[M]$ such that $XA^E = H = YB^E$ has rank kn, then the matrix P, unimodular in $[M]$, such that $PA = B$, is given by both $P^E = Y^{-1}X$ and $P^E = B^E(A^E)^{-1}$.*

7. The singular case with principal column class. If the matrices A and B of $[M]$ are singular, the fact that A^E and B^E have the same Hermite form guarantees the existence of a matrix P, unimodular in $[M]$, such that $PA = B$, but a direct construction of the transforming matrix P seems to be attended with a number of difficulties, as forewarned in **1**.

Using the notion of the division of the ideals of $[F]$ into classes of equivalent ideals, we can, after a few preliminary definitions, state Steinitz' elegant theorem, exactly descriptive of the situation.

Let $A^{(m)}$ indicate the *m-th adjugate* of A, i. e., the matrix of order $\binom{n}{m}$ formed by arranging all the determinants of order m that can be formed from A in lexicographic order as to rows and columns. If A is of rank r, then $A^{(r)}$ is of rank 1; hence the columns of $A^{(r)}$ are proportional; hence

the ideals determined by the different columns of $A^{(r)}$ belong to the same class of ideals, described as the *column class* of A.

If A is in $[M]$ and if $n - b$ is the maximum number of zero rows to be found in any matrix QA, where Q is unimodular in $[M]$, then the matrix A is said to possess an *irreducible basis* of b rows.

The *principal class* is the class of ideals of $[F]$ consisting of all the principal ideals of $[F]$.

In terms of these definitions the theorem of Steinitz is as follows:

If the matrix A of $[M]$ is of rank r, then A has an irreducible basis of $b = r$ or $b = r + 1$ rows, according as the column class of A is or is not the principal class.

Only an outline of the proof will be presented here.[6] A row q of n elements of $[F]$ is said to belong to the left null-space of A if $qA = 0$. If the rank r of A is such that $r \leqq n - 2$ so that the left null-space of A has at least two independent rows, then it can be proved that the left null-space of A contains a row of relatively prime numbers of $[F]$; it can be proved that this latter row can be augmented to a matrix Q unimodular in $[M]$ and such that QA has at least one zero row. Continuing in this manner, we have the result that in all cases the irreducible basis has either $b = r$ or $b = r + 1$ rows, the proof being obvious in the omitted cases $r = n$ and $r = n - 1$. The proof is concluded by identifying the case $b = r$ with a matrix A whose column class is the principal class.

In some of these operations the method of enlarged matrices is helpful. For example, in considering the left null-space of A, we can use $XA^E = H$; for from any one of the $k(n - r)$ rows of X with elements in $[Ra]$ which produce a zero row of H, using the lemma of **2**, we can immediately derive a row with elements in $[F]$ belonging to the left null-space of A. It does not follow, however, that two distinct rows of X which are, of course, independent in Ra, will have derived rows independent in F.

We can also give a method based on enlarged matrices for deciding whether the elements of a row are relatively prime; for we can give an explicit construction for the canonical minimal basis of any ideal of $[F]$. Consider, for example, the ideal $(a_1, \cdots, a_n)$. If the Hermite form of the kn-by-k column vector formed from $S(a_1), \cdots, (Sa_n)$ has in its last k-by-k block the elements h_{ij}, with $h_{ij} = 0$ if $j > i$, then $h_1 = h_{11}e_1$, $h_2 = h_{21}e_1 + h_{22}e_2, \cdots, h_k = h_{k1}e_1 + \cdots + h_{kk}e_k$ form the unique canonical

[6] For example, see Franz, *loc. cit.*, Hilfsatz 3, Satz 5, Satz 6.

minimal basis for the ideal.[7] In particular, if $(a_1, \cdots, a_n) = (1)$, then the last k-by-k block of the Hermite form must be $S(1) = I_k$. Furthermore, there is a direct construction for the numbers $p_1, \cdots, p_n$ of $[F]$ such that $p_1a_1 + \cdots + p_na_n = 1$. For if P is the matrix unimodular in $[Ra]$ such that

$$P \begin{pmatrix} S(a_1) \\ \cdot \\ \cdot \\ \cdot \\ S(a_n) \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ I \end{pmatrix}$$

then from the $k(n-1)+1$-st row of P we may derive, in the manner of the lemma of **2**, the values of $p_1, \cdots, p_n$.

Unfortunately the Steinitz process of augmenting a row of relatively prime numbers of $[F]$ to a unimodular matrix of $[M]$ is more complicated for the general algebraic domain $[F]$ than it is for the well-known case of the rational domain $[Ra]$, and it does not seem that the method of enlarged matrices is particularly helpful in this fundamental step.

As a corollary to this Steinitz theorem we observe that if the domain $[F]$ is a principal ideal ring, then the column class of each matrix A must be the principal class and hence for each A we find $b = r$, the familiar situation in $[Ra]$.

By the method of enlarged matrices we have noted how to construct for any ideal a minimal basis of k elements, i. e., a basis with respect to $[Ra]$. As a further corollary to the Steinitz theorem we note that an ideal can be written as a column vector of rank 1, and therefore has an irreducible basis, i. e., a basis with respect to $[F]$, of one or two elements, according as the ideal is principal or non-principal.

In conclusion to this section we have the following result:

THEOREM III. *If A and B are singular matrices of $[M]$ of rank r whose enlarged matrices have the same Hermite form, whose column class is the principal class, and which are in preferred form, then the matrix P unimodular in $[M]$ such that $PA = B$ is given by $P = R^{-1}P'Q$. The matrices Q and R are unimodular in $[M]$ such that the last r rows of $QA = A'$ and $RB = B'$ are the only non-zero rows. The matrix P' unimodular in $[M]$ is determined by $(P')^E = Y^{-1}X$ where $X(A')^E = H = Y(B')^E$ with matrices as follows:*

[7] MacDuffee, *loc. cit.* in [1].

$$(A')^E = \begin{pmatrix} 0 & 0 \\ A^E{}_2 & A^E{}_1 \end{pmatrix}, \quad (B')^E = \begin{pmatrix} 0 & 0 \\ B^E{}_2 & B^E{}_1 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 \\ H_2 & H_1 \end{pmatrix}$$

$$X = \begin{pmatrix} I & 0 \\ 0 & X_1 \end{pmatrix}, \quad Y = \begin{pmatrix} I & 0 \\ 0 & Y_1 \end{pmatrix}, \quad (P')^E = \begin{pmatrix} I & 0 \\ 0 & P^E{}_1 \end{pmatrix},$$

where $P^E{}_1$ *is given by both* $P^E{}_1 = Y^{-1}{}_1 X_1$ *and* $P^E{}_1 = B^E{}_1 (A^E{}_1)^{-1}$.

8. The singular case with a non-principal column class. If the matrices A and B of $[M]$ are singular of rank r with a non-principal column class, then there exist unimodular matrices Q and R of $[M]$ such that the last $r+1$ rows of $QA = A'$ and $RB = B'$ are the only non-zero rows. If $X'(A')^E = H = Y'(B')^E$, then we succeed, as described in **5**, in constructing matrices U and V of $[M]$ such that $UA' = B'$ and $VB' = A'$. The theorem of Steinitz, given in **4**, then guarantees the existence of a matrix P', unimodular in $[M]$, such that $P'A' = B'$. In the cases of matrices that are non-singular or are singular with a principal column class the discussions of **6** and **7** provide a construction for P'. But in the case here under discussion a very different attack seems to be required; we shall outline Steinitz' proof and indicate a number of steps in which the use of enlarged matrices is helpful. Having found P', we can construct $P = R^{-1}P'Q$ unimodular in $[M]$ and such that $PA = B$.

If A and B are in preferred form, then

$$A' = \begin{pmatrix} 0 & 0 \\ A_2 & A_1 \end{pmatrix}, \qquad B' = \begin{pmatrix} 0 & 0 \\ B_2 & B_1 \end{pmatrix},$$

where A_1 and B_1 are $r+1$-by-$r+1$ and of rank r. Therefore

$$Y' = \begin{pmatrix} I & 0 \\ 0 & X_1 \end{pmatrix}, \qquad Y' = \begin{pmatrix} I & 0 \\ 0 & Y_1 \end{pmatrix},$$

where X_1 and Y_1 are $k(r+1)$-by-$k(r+1)$ and unimodular in $[Ra]$. Matrices U_1 and V_1 can therefore be derived from $Y^{-1}{}_1 X_1$ and $X^{-1}{}_1 Y_1$, respectively, as explained in **5**, so that

$$U = \begin{pmatrix} I & 0 \\ 0 & U_1 \end{pmatrix}, \qquad V = \begin{pmatrix} I & 0 \\ 0 & V_1 \end{pmatrix}.$$

The essential problem reduces to being given matrices A_1 and B_1 of order $r+1$ and of rank r with elements in $[F]$ and mutually left divisible, since $U_1 A_1 = B_1$ and $V_1 B_1 = A_1$, and being asked to prove that A_1 and B_1 are left associated, i. e., that there exists a matrix P_1 of order $r+1$ and unimodular in $[F]$ such that $P_1 A_1 = B_1$. For having found P_1 we may take

$$P' = \begin{pmatrix} I & 0 \\ 0 & P_1 \end{pmatrix}.$$

Since A_1 is of order $r+1$ and rank r, the left null-space of A_1 has a basis of one row $w_1 = w_{11}, \cdots, w_{1,r+1}$. Since the first k rows of $X_1 A_1^E = H_1$ are zero rows, it is easy to derive, as in the lemma of **2**, a row with elements in $[F]$ to serve as w_1. Let T_1 indicate the ideal determined by the elements of w_1. Any other row $w_2 = w_{21}, \cdots, w_{2,r+1}$ in the left null-space of A_1 and with elements in $[F]$ determines an ideal T_2 in the same ideal class as T_1 because the row w_2 is proportional to the row w_1. Since in every ideal class there exist ideals relatively prime to any given ideal,[8] the row w_2 can be determined with elements in $[F]$ so that $(T_1, T_2) = (1)$. It is then possible to express any other row $w_3 = w_{31}, \cdots, w_{3,r+1}$ of the left null-space of A_1 with elements in $[F]$ in the form $w_3 = m_1 w_1 + m_2 w_2$ where m_1 and m_2 are in $[F]$. For in terms of ideal factors we have $w_{1i} = T_1 W_i$, $w_{2i} = T_2 W_i$, $(w_{1i}, w_{2i}) = (T_1 W_i, T_2 W_i) = (T_1, T_2) W_i = W_i$; hence $w_{3i} = T_3 W_i$ which is in the ideal W_i can be written $w_{3i} = m_1 w_{1i} + m_2 w_{2i}$ where m_1 and m_2 are in $[F]$. The same coefficients m_1 and m_2 serve for $i = 1, \cdots, r+1$ because the rows w_1, w_2, w_3 are proportional.

Consider any row w with elements in $[F]$, but not necessarily in the left null-space of A_1. From $wA_1 = wV_1B_1 = wV_1U_1A_1$ it follows that the row $w - wV_1U_1$ is in the left null-space of A_1 and can be written $w - wV_1U_1 = m_1 w_1 + m_2 w_2$. Hence w is a linear combination with coefficients in $[F]$ of the rows w_1 and w_2 and the rows of U_1. If this result is applied, successively, to the rows of the identity matrix I of order $r+1$, we see that there exists a matrix Z which is $r+1$-by-$r+3$ with elements in $[F]$ such that

$$I = Z \begin{pmatrix} U_1 \\ w_1 \\ w_2 \end{pmatrix}.$$

In terms of the $r+1$-st adjugates we find

$$1 = I^{(r+1)} = Z^{(r+1)} \begin{pmatrix} U_1 \\ w_1 \\ w_2 \end{pmatrix}^{(r+1)};$$

so that (1) is the greatest common ideal divisor of the determinants of the $r+1$-st order that can be formed from $\begin{pmatrix} U_1 \\ w_1 \\ w_2 \end{pmatrix}$. Since w_1 and w_2 are pro-

[8] Steinitz, *loc. cit.*, Part I, 1.

portional many of these $r+1$-st order determinants are zero, and the result may be written

$$(4) \qquad (1) = (d(U_1), d(M_1), \cdots, d(M_{r+1}), d(N_1), \cdots, d(N_{r+1})),$$

where M_i is the matrix formed by replacing the i-th row of U_1 by w_1, and N_i is the matrix formed by replacing the i-th row of U_1 by w_2. If $U_1 = (u_{ij})$, let U_{ij} denote the cofactor of u_{ij}, so that $d(M_i) = \sum_j w_{1j}U_{ij}$ and $d(N_i) = \sum_j w_{2j}U_{ij}$. Using these formulas and well-known properties of the cofactors, we can easily establish the following results:

$$(5) \qquad \begin{aligned} \sum_i d(M_i)u_{it} &= w_{1t}d(U_1), \\ \sum_i d(N_i)u_{it} &= w_{2t}d(U_1), \end{aligned} \qquad t = 1, \cdots, r+1.$$

Suppose that $d(M_1), \cdots, d(M_{r+1}), d(N_1), \cdots, d(N_{r+1})$ have a common ideal divisor T. From (4) it follows that either $T = (1)$ or that T does not divide $d(U_1)$. In the latter case it follows from (5) that T divides w_{1i} and w_{2i} for $i = 1, \cdots, r+1$. But $(w_{11}, \cdots, w_{1,r+1}, w_{21}, \cdots, w_{2,r+1}) = (T_1, T_2) = (1)$; hence again $T = (1)$. Therefore

$$(1) = (d(M_1), \cdots, d(M_{r+1}), d(N_1), \cdots, d(N_{r+1})).$$

Using the method of enlarged matrices, described in **7**, for determining a minimal basis of an ideal, we have an explicit method for determining the numbers $a'_1, \cdots, a'_{r+1}, b'_1, \cdots, b'_{r+1}$ of $[F]$ such that

$$(6) \quad a'_1d(M_1) + \cdots + a'_{r+1}d(M_{r+1}) + b'_1d(N_1) + \cdots + b'_{r+1}d(N_{r+1}) = 1.$$

Let us define $a_i = (1 - d(U_1))a'_i$, $b_i = (1 - d(U_1))b'_i$, $i = 1, \cdots, r+1$; and

$$(7) \qquad P_1 = U_1 + \begin{pmatrix} a_1w_1 \\ \cdot \\ \cdot \\ \cdot \\ a_{r+1}w_1 \end{pmatrix} + \begin{pmatrix} b_1w_2 \\ \cdot \\ \cdot \\ \cdot \\ b_{r+1}w_2 \end{pmatrix}.$$

Well-known properties of determinants enable us to expand

$$d(P_1) = d(U_1) + \sum_i a_id(M_i) + \sum_i b_id(N_i);$$

by virtue of (6) we conclude that $d(P_1) = 1$; hence P_1 is a matrix unimodular in $[F]$. Furthermore, since $w_1A_1 = 0 = w_2A_1$, we find $P_1A_1 = U_1A_1 = B_1$. Hence A_1 and B_1 are left associates as was to be proved.

Those steps of the above proof which would be essential in the construction of the transforming matrix can be summarized, as follows:

THEOREM IV. *If A and B are singular matrices of $[M]$ of rank r whose enlarged matrices have the same Hermite form, whose column class is not the principal class, and which are in preferred form, then the matrix P unimodular in $[M]$ such that $PA = B$ is given by $P = R^{-1}P'Q$. The matrices Q and R are unimodular in $[M]$ such that the last $r+1$ rows of $QA = A'$ and $RB = B'$ are the only non-zero rows. From $X'(A')^E = H = Y'(B')^E$ determine X_1 and Y_1. From $Y_1^{-1}X_1$ derive U_1 such that $U_1A_1 = B_1$. From one of the first k rows of X_1 derive a row w_1 such that $w_1A_1 = 0$. Determine a row w_2 such that $w_2A_1 = 0$ and $(T_1, T_2) = (1)$, where $T_i = (w_{i1}, \cdots, w_{i,r+1})$. Compute $d(M_1), \cdots, d(N_{r+1})$, where M_i and N_i are the matrices obtained from U_1 by replacing the i-th row by w_1 and w_2, respectively. Using enlarged matrices, from*

$$N \begin{pmatrix} S(d(M_1)) \\ \cdot \\ \cdot \\ \cdot \\ S(d(N_{r+1})) \end{pmatrix} = S(1)$$

derive the numbers $a'_1, \cdots, b'_{r+1}$ satisfying (6). Construct P_1 as in (7) and take

$$P' = \begin{pmatrix} I & 0 \\ 0 & P_1 \end{pmatrix}.$$

MICHIGAN STATE COLLEGE,
EAST LANSING, MICHIGAN.

EXTENSIONS OF HARMONIC TRANSFORMATIONS.*

By EDWARD KASNER and JOHN DE CICCO.

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1. Harmonic transformations. We shall study the transformation theory of differential elements of third order, all of which pass through a fixed point, which is induced by the infinite set of harmonic transformations.[1] Also we shall consider the transformation theory of third order differential elements under the group of arbitrary point transformations. From this work, a comparison of the two theories may be deduced.

We shall term a correspondence T of the (x, y)-plane a *harmonic transformation* if it is defined by the equations

$$X = \phi(x, y), \qquad Y = \psi(x, y), \tag{1}$$

where the jacobian

$$J = \phi_x\psi_y - \phi_y\psi_x \neq 0, \tag{2}$$

such that the components ϕ and ψ obey the Laplace equation

$$\phi_{xx} + \phi_{yy} = 0, \qquad \psi_{xx} + \psi_{yy} = 0, \tag{3}$$

but otherwise are not interrelated in any way whatsoever.

Harmonic transformations should not be confused with conformal correspondences. Conformal maps are those for which the components ϕ and ψ obey the direct or reverse Cauchy-Riemann equations

$$\phi_x = \pm\,\psi_y, \qquad \phi_y = \mp\,\psi_x. \tag{4}$$

The components ϕ and ψ of a harmonic transformation do not satisfy any differential conditions of first order in general, and therefore a harmonic transformation is not usually conformal. Thus the conformal group is a proper subset of the infinite set of harmonic transformations.

The group of arbitrary point transformations induces the general projective group of three parameters between the pencils of lineal elements whose vertices are at the points corresponding under a transformation T. *The infinite set (H) of harmonic transformations induces the same three-*

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[1] Kasner and De Cicco, "Theory of harmonic transformations," *Proceedings of the National Academy of Sciences*, vol. 33 (1947), pp. 20-23.

parameter projective group. The conformal group induces a one-parameter group, namely, the group of rotations.

By the group of arbitrary point transformations, there corresponds an eight-parameter group G_8 between the bundles of differential elements of second order whose vertices are at the points corresponding under the transformation T. Kasner has presented a complete study of this group,[2] obtaining all the possible invariants and also classifying all the differential equations of second order which are algebraic in the derivatives by means of this group G_8. *By the infinite set (H) of harmonic transformations, there corresponds the same eight-parameter group G_8 between the bundles of second order differential elements.* By the conformal group, there corresponds a four-parameter group G_4.

The group of arbitrary point transformations induces on the differential element of third order a fifteen-parameter group G_{15}. This group G_{15} is equivalent to projective geometry on a three-dimensional manifold M_5 of fifth degree embedded in projective space R_{11} of eleven dimensions. By means of this fifteen-parameter group G_{15}, we obtain a classification of differential equations of third order which are algebraic in the derivatives.

The infinite set of harmonic transformations induces on the differential elements of third order a twelve-parameter set S_{12} of transformations. This set S_{12} is not a group. Thus the first essential difference in the extensions of the group of arbitrary point transformations and those of the infinite set of harmonic transformations is in the third order of differentiation.

It is noted that by the conformal group, there corresponds a six-parameter group G_6 of transformations of third order differential elements, which we have studied elsewhere.[3].

2. Transformation theory of differential elements of second order. In order to simplify our work, we shall use the minimal coordinates $u = x + iy$

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[2] Kasner, "The geometry of differential elements of the second order with respect to the group of all point transformations," *American Journal of Mathematics*, vol. 28 (1906), pp. 203-213.

[3] Kasner and De Cicco, "Conformal geometry of third order differential elements," *Revista de la Universidad Nacional de Tucuman (Argentina)*, vol. 2 (1941), pp. 51-58. Also De Cicco, "Equilong geometry of third order differential elements," *National Mathematics Magazine*, vol. 19 (1945). Kasner, "Conformal geometry," *Proceedings of the Fifth International Congress of Mathematicians, Cambridge*, vol. 2 (1912), pp. 81-90. See De Cicco, "Differential geometry in the Kasner plane," *American Mathematical Monthly*, vol. 53, pp. 305-313, where an extensive bibliography on the Conformal geometry of the horn angle is given.

and $v = x - iy$. An arbitrary point transformation T may be written in the form

$$U = U(u, v), \qquad V = V(u, v), \tag{5}$$

where the jacobian is

$$J = U_u V_v - U_v V_u \neq 0. \tag{6}$$

The transformation T as defined by (5) is harmonic if and only if each of the components $U(u, v)$ and $V(u, v)$ obey the modified Laplace equation

$$U_{uv} = 0, \qquad V_{uv} = 0. \tag{7}$$

The transformation T is conformal if and only if $U_v = V_u = 0$, or $U_u = V_v = 0$.

For convenience, we introduce the letters (p, q, r) to denote the first three total derivatives of v with respect to u. That is, let $p = dv/du$, $q = d^2v/du^2$, $r = d^3v/du^3$. Thus (u, v, p) denotes a differential element of first order, (u, v, p, q) indicates a second order differential element, and (u, v, p, q, r) represents a differential element of third order.

The first extension of the transformation T as defined by (5) is

$$P = \frac{b_0 + b_1 p}{a_0 + a_1 p}, \tag{8}$$

where

$$\begin{gathered} a_0 = U_u, \qquad a_1 = U_v, \qquad b_0 = V_u, \qquad b_1 = V_v, \\ J = a_0 b_1 - a_1 b_0 = U_u V_v - U_v V_u \neq 0. \end{gathered} \tag{9}$$

Thus the group of arbitrary point transformations induces the general projective group of three-parameters between the pencils of lineal elements whose vertices are at the points corresponding under the transformation T. *This statement is valid even for the infinite set of harmonic transformations.* Of course under the conformal group, equation (8) defines a one-parameter group G_1, namely, the group of rotations.

The second extension of the transformation T is

$$Q = \frac{c_0 + c_1 p + c_2 p^2 + c_3 p^3 + Jq}{(a_0 + a_1 p)^3}, \tag{10}$$

where

$$\begin{gathered} c_0 = U_u V_{uu} - V_u U_{uu}, \qquad c_1 = -J_u + 3(U_u V_{uv} - V_u U_{uv}), \\ c_2 = J_v + 3(U_v V_{uv} - V_v U_{uv}), \qquad c_3 = U_v V_{vv} - V_v U_{vv}. \end{gathered} \tag{11}$$

The group of arbitrary point transformations induces a group G_8 of eight parameters defined by equations (8) and (10) between the bundles of differ-

ential elements of second order whose vertices are at the points corresponding under the transformation T. It is observed by (11) that *the infinite set of harmonic transformations induces the same eight-parameter group G_8 between the bundles of second order differential elements.* Under the conformal group, equations (8) and (10) define merely a four-parameter group G_4.

Kasner has presented a complete study of this eight-parameter group G_8. We shall consider briefly some of his results of which extensions are given in the next section.

A differential element of second order (which passes through a fixed point) is defined by the ordered number pair (p, q) and can be represented as a point in four-dimensional projective space R_4 with homogeneous coordinates

$$x_0 : x_1 : x_2 : x_3 : x_4 = 1 : p : p^2 : p^3 : q. \tag{12}$$

The ∞^2 points in R_4 representing the ∞^2 differential elements of second order constitute the manifold defined by

$$x_0x_2 - x_1^2 = 0, \qquad x_0x_3 - x_1x_2 = 0, \qquad x_1x_3 - x_2^2 = 0. \tag{13}$$

This is a cubic cone S_3 with its vertex at the point $(0, 0, 0, 0, 1)$. Its section in the space $x_4 = 0$ is a twisted cubic.

The geometry of a bundle of differential elements of second order in the plane with respect to the group of arbitrary point transformaition is equivalent to projective geometry on a cubic cone S_3 in a space R_4 of four dimensions.

The above principle establishes a classification of differential equations of second order which are algebraic in the derivatives. Kasner defined the *rank* n of such a differential equation of second order to be the degree n of the corresponding algebraic manifold in R_4. This rank n is invariant under the group of arbitrary point transformations.

In particular, any second order differential equation of the *first rank* is

$$v'' = Av'^3 + Bv'^2 + Cv' + D, \tag{14}$$

where the coefficients (A, B, C, D) are arbitrary functions of (u,v). This is the cubic type that has been studied extensively by Lie, R. Liouville, Tresse, Kasner, Wilczynski, and Douglas.

The second order differential equations of the *second rank* are

$$(15)\qquad \begin{aligned} &Av''^2 + (B_0 + B_1v' + B_2v'^2 + B_3v'^3)v'' \\ &\quad + (C_0 + C_1v' + C_2v'^2 + C_3v'^3 + C_4v'^4 + C_5v'^5 + C_6v'^6) = 0, \end{aligned}$$

where the coefficients are arbitrary functions of (u, v).

Equality of rank is necessary but of course not sufficient for the equivalence of second order differential equations under the group of arbitrary point transformations.

3. Transformation theory of differential elements of third order under the group of arbitrary point transformations. The third extension of the transformation T defined by the general equations (5) is

$$(16)\qquad \begin{aligned} R = &\frac{d_0 + d_1p + d_2p^2 + d_3p^3 + d_4p^4 + d_5p^5}{(a_0 + a_1p)^5} \\ &+ \frac{(h_0 + h_1p + h_2p^2)q - 3a_1Jq^2 + (a_0 + a_1p)Jr}{(a_0 + a_1p)^5}, \end{aligned}$$

where

$$(17)\qquad \begin{aligned} d_0 &= U_uc_{0u} - 3U_{uu}c_0, \\ d_1 &= U_u(c_{0v} + c_{1u}) + U_vc_{0u} - 3U_{uu}c_1 - 6U_{uv}c_0, \\ d_2 &= U_u(c_{1v} + c_{2u}) + U_v(c_{0v} + c_{1u}) - 3U_{uu}c_2 - 6U_{uv}c_1 - 3U_{vv}c_0, \\ d_3 &= U_u(c_{2v} + c_{3u}) + U_v(c_{1v} + c_{2u}) - 3U_{uu}c_3 - 6U_{uv}c_2 - 3U_{vv}c_1, \\ d_4 &= U_uc_{3v} + U_v(c_{2v} + c_{3u}) - 6U_{uv}c_3 - 3U_{vv}c_2, \\ d_5 &= U_vc_{3v} - 3U_{vv}c_3, \\ h_0 &= U_u(c_1 + J_u) - 3JU_{uu} - 3c_0U_v, \\ h_1 &= U_u(J_v + 2c_2) + U_v(J_u - 2c_1) - 6U_{uv}J, \\ h_2 &= 3U_uc_3 + U_v(J_v - c_2) - 3U_{vv}J. \end{aligned}$$

Upon writing the formulas for the partial derivatives c_{0u}, c_{1u}, c_{2v}, c_{3v}, we find that the resulting equations can be solved for U_{uuu}, V_{uuu}, U_{vvv}, V_{vvv}. This demonstrates that c_{0u}, c_{1u}, c_{2v}, c_{3v} are independent expressions. By the first, second, fifth, and sixth of equations (17), we find

$$(18)\qquad \begin{aligned} &c_{0u} = (1/a_0)(d_0 + 3c_0U_{uu}), \qquad c_{3v} = (1/a_1)(d_5 + 3c_3U_{vv}), \\ &c_{0v} + c_{1u} = d_1/a_0 - (1/a_0^2)[a_1d_0 + 3(a_1c_0 - a_0c_1)U_{uu} - 6a_0c_0U_{uv}], \\ &c_{2v} + c_{3u} = d_4/a_1 - (1/a_1^2)[a_0d_5 + 3(a_0c_3 - a_1c_2)U_{vv} - 6a_1c_3U_{uv}]. \end{aligned}$$

Substituting these into the third and fourth of equations (17) and then eliminating U_{uu}, U_{vv}, V_{uu}, V_{vv}, by means of equations (11), we obtain

$$
\begin{aligned}
&J(a_0^2d_2 - a_0a_1d_1 + a_1^2d_0)\\
&\quad - 3(a_0^3c_0c_3 + a_0^3c_1c_2 - 2a_0^2a_1c_0c_2 - a_0^2a_1c_1^2 + 2a_0a_1^2c_0c_1 - a_1^3c_0^2)\\
&\quad = a_0^3[J(c_{2u} + c_{1v}) + 6(a_1c_1 - a_0c_2)V_{uv} - 6(b_1c_1 - b_0c_2)U_{uv}],\\
&J(a_1^2d_3 - a_0a_1d_4 + a_0^2d_5)\\
&\quad - 3(a_0^3c_3^2 - 2a_0^2a_1c_2c_3 + a_0a_1^2c_2^2 + 2a_0a_1^2c_1c_3 - a_1^3c_1c_2 - a_1^3c_0c_3)\\
&\quad = a_1^3[J(c_{2u} + c_{1v}) + 6(a_1c_1 - a_0c_2)V_{uv} - 6(b_1c_1 - b_0c_2)U_{uv}].
\end{aligned}
\tag{19}
$$

From these equations, we deduce the result

$$
\begin{aligned}
&J[a_0^5d_5 - a_0^4a_1d_4 + a_0^3a_1^2d_3 - a_0^2a_1^3d_2 + a_0a_1^4d_1 - a_1^5d_0]\\
&\quad = 3[a_0^3c_3 - a_0^2a_1c_2 + a_0a_1^2c_1 - a_1^3c_0]^2.
\end{aligned}
\tag{20}
$$

Upon eliminating U_{uu}, U_{vv}, V_{uu}, V_{vv} by means of equations (11) from the last three of equations (17), we find

$$
\begin{aligned}
h_0 &= -6a_1c_0 + 3a_0c_1 - 3a_0(a_0V_{uv} - b_0U_{uv}),\\
h_1 &= 3(a_0c_2 - a_1c_1) - 3(a_0b_1 - a_1b_0)U_{uv},\\
h_2 &= 6a_0c_3 - 3a_1c_2 + 3a_1(a_1V_{uv} - b_1U_{uv}).
\end{aligned}
\tag{21}
$$

From these, we may eliminate the derivatives U_{uv} and V_{uv}. Thus we get the relation

$$
a_0^2h_2 + a_0a_1h_1 + a_1^2h_0 = 6[a_0^3c_3 - a_0^2a_1c_2 + a_0a_1^2c_1 - a_1^3c_0]. \tag{22}
$$

THEOREM. *The group of arbitrary point transformation induces on the differential elements of third order the group of transformations*

$$
\begin{aligned}
P &= \frac{b_0 + b_1p}{a_0 + a_1p}, \qquad Q = \frac{c_0 + c_1p + c_2p^2 + c_3p^3 + Jq}{(a_0 + a_1p)^3},\\
R &= \frac{d_0 + d_1p + d_2p^2 + d_3p^3 + d_4p^4 + d_5p^5}{(a_0 + a_1p)^5}\\
&\quad + \frac{(h_0 + h_1p + h_2p^2)q - 3a_1Jq^2 + (a_0 + a_1p)Jr}{(a_0 + a_1p)^5},
\end{aligned}
\tag{23}
$$

where the seventeen parameters satisfy the two relationships

$$
\begin{aligned}
&J[a_0^5d_5 - a_0^4a_1d_4 + a_0^3a_1^2d_3 - a_0^2a_1^3d_2 + a_0a_1^4d_1 - a_1^5d_0]\\
&\quad = 3[a_0^3c_3 - a_0^2a_1c_2 + a_0a_1^2c_1 - a_1^3c_0]^2,\\
&a_0^2h_2 + a_0a_1h_1 + a_1^2h_0 = 6[a_0^3c_3 - a_0^2a_1c_2 + a_0a_1^2c_1 - a_1^3c_0].
\end{aligned}
\tag{24}
$$

This induced group is therefore of fifteen essential parameters.

A differential element of third order (which passes through a fixed point) may be denoted by the ordered number triplet (p, q, r), and can be depicted as a point in eleven-dimensional projective space R_{11} with homogeneous coordinates

$$(25)\quad \begin{aligned} &x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6 : x_7 : x_8 : x_9 : x_{10} : x_{11} \\ &= 1 : p : p^2 : p^3 : p^4 : p^5 : q : pq : p^2q : r : pr. \end{aligned}$$

The ∞^3 points in R_{11} representing the ∞^3 differential elements of third order form the manifold defined by the system of eight independent equations

$$(26)\quad \begin{aligned} &x_0x_2 - x_1^2 = 0, \quad x_1x_3 - x_2^2 = 0, \quad x_2x_4 - x_3^2 = 0, \quad x_3x_5 - x_4^2 = 0, \\ &x_4x_7 - x_5x_6 = 0, \quad x_6x_8 - x_7^2 = 0, \quad x_0x_9 - x_6^2 = 0, \quad x_8x_{11} - x_7x_{10} = 0. \end{aligned}$$

This is a three-dimensional algebraic manifold M_5 of the fifth degree.

The geometry of a bundle of ∞^3 differential elements of third order in the plane with respect to the group of arbitrary point transformations is equivalent to projective geometry on a three-dimensional algebraic manifold M_5 of the fifth degree in a space R_{11} of eleven dimensions.

By this principle, we may now define the *rank* n of a differential equation of third order which is algebraic in the derivatives (v', v'', v''') to be the degree n of the corresponding algebraic variety in R_{11}. This rank n is invariant under the group of arbitrary point transformations of the plane of minimal coordinates (u, v).

In particular, any third order differential equation of the *first rank* is

$$(27)\quad \begin{aligned} &(A_0 + A_1v')v''' + B_0v''^2 + (C_0 + C_1v' + C_2v'^2)v'' \\ &+ (D_0 + D_1v' + D_2v'^2 + D_3v'^3 + D_4v'^4 + D_5v'^5) = 0. \end{aligned}$$

where the twelve coefficients are arbitrary functions of (u, v). This type includes the differential equation of the ∞^3 Minding geodesic circles of any surface Σ when Σ is represented on the plane in any arbitrary point-to-point fashion,[4] and also the differential equation of the ∞^3 dynamical trajectories of any positional field of force.[5]

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[4] Kasner and De Cicco, "Families of curves conformally equivalent to circles," *Transactions of the American Mathematical Society*, vol. 49 (1941), pp. 378-391. Also De Cicco, "Equilong maps of the ∞^3 circles," *Transactions of the American Mathematical Society*, vol. 59 (1946), pp. 42-53.

[5] Kasner, "Differential-geometric aspects of dynamics," *American Mathematical Society Colloquium Publications*, vol. 3, 1913, 1934. Also see a series of papers in *Transactions of the American Mathematical Society*, vols. 7-11 (1906-1910).

Finally see Terracini, "Sobre la equacion diferencial $y''' = G(x, y, y')y'' + H(x, y, y')y''^2$," *Revista de Matematicas de la Universidad de Tucuman (Argentina)*, vol. 2 (1941), pp. 245-329.

4. Transformation theory of differential elements of third order under the infinite set (H) of harmonic transformations. Finally we consider the case of the infinite set (H) of harmonic transformations T. For any such correspondence T, we have $U_{uv} = 0$ and $V_{uv} = 0$. It follows that $c_{2u} + c_{1v} = 0$. Thus by studying equations (19) and (21), we discover the following fundamental proposition.

THEOREM. *The infinite set (H) of harmonic transformations induces on the differential elements of third order the set of transformations* (23) *where the seventeen parameters satisfy the five relationships*

$$J(a_0^2 d_2 - a_0 a_1 d_1 + a_1^2 d_0)$$
$$= 3(a_0^3 c_0 c_3 + a_0^3 c_1 c_2 - 2a_0^2 a_1 c_0 c_2 - a_0^2 a_1 c_1^2 + 2a_0 a_1^2 c_0 c_1 - a_1^3 c_0^2),$$

(28)
$$J(a_1^2 d_3 - a_0 a_1 d_4 + a_0^2 d_5)$$
$$= 3(a_0^3 c_3^2 - 2a_0^2 a_1 c_2 c_3 + a_0 a_1^2 c_2^2 + 2a_0 a_1^2 c_1 c_3 - a_1^3 c_1 c_2 - a_1^3 c_0 c_3),$$
$$h_0 = -6a_1 c_0 + 3a_0 c_1, \qquad h_1 = 3a_0 c_2 - 3a_1 c_1, \qquad h_2 = 6a_0 c_3 - 3a_1 c_2.$$

This induced set S_{12} involves twelve essential parameters.

It is remarked that under the group of conformal transformations, the induced transformations (23) form a six-parameter group G_6 as we have shown in the first article of footnote 3.

Thus the infinite set (H) of harmonic transformations induces between the bundles of differential elements of first, second, and third orders respectively, the following sets of transformations:

(1). The three-parameter projective group G_3.

(2). The eight-parameter group G_8.

(3). The twelve-parameter set S_{12}.

In the case of differential elements of the first and second orders, the induced sets are identical with those induced by the larger group of arbitrary point transformations. However, in the theory of differential elements of third order, the infinite set (H) of harmonic transformations induces a set S_{12} of *twelve-parameters,* which is a proper subset of the *fifteen-parameter group* G_{15} induced by the group of all point transformations.

COLUMBIA UNIVERSITY, NEW YORK,
AND
ILLINOIS INSTITUTE OF TECHNOLOGY, CHICAGO.

ON CONVERGENCE OF THE PRODUCT OF BASIC SETS OF POLYNOMIALS.*

By M. N. GHABBOUR.

The subject of basic sets of polynomials has been studied principally by J. M. Whittaker and B. Cannon.[1] For such a set $\{p_n(z)\}$ any polynomial, and in particular the polynomial z^n, admits a unique finite representation of the form

$$z^n = \pi_{n0}p_0(z) + \pi_{n1}p_1(z) + \pi_{n2}p_2(z) + \cdots, \tag{1}$$

Thus any set $\{p_n(z)\}$ of polynomials, where $p_n(z)$ is of degree n, is basic and is said to be *simple*. Generally if the number N_n of nonzero coefficients in (1) is such that $N_n^{1/n}$ tends to 1 as n tends to infinity the set $\{p_n(z)\}$ is called a *Cannon set*. Associated with any function $f(z)$, regular about the origin, there is a *basic series* $\Sigma\Pi_n f(0)p_n(z)$, where the operators (Π_n) are given by

$$\Pi_n = \pi_{0n} + \pi_{1n}(d/dz) + (1/2!)\pi_{2n}(d^2/dz^2) + \cdots \quad (n = 0, 1, 2, \cdots).$$

This series is said to *represent* $f(z)$ in $|z| \leq R$, where the function is regular, if it converges uniformly to $f(z)$ in $|z| \leq R$. The basic series of the set $\{p_n(z)\}$ is said to be *effective in* $|z| \leq R$, if it represents, in $|z| \leq R$, every function which is regular there. This is equivalent to the simpler statement that the set $\{p_n(z)\}$ is *effective in* $|z| \leq R$. Let

$$\delta_n(R) = \sum_i |\pi_{ni}| A_i(R), \tag{2}$$

where $A_n(R)$ is the maximum modulus of $p_n(z)$ in $|z| \leq R$. Cannon (1) has[2] shewn that a necessary and sufficient condition for a Cannon set $\{p_n(z)\}$ to be effective in $|z| \leq R$, is that

$$\delta(R) = \overline{\lim_{n\to\infty}} \{\delta_n(R)\}^{1/n} = R, \tag{3}$$

for any value of $R > 0$. Suppose that $p_n(z) = p_{n0} + p_{n1}z + p_{n2}z^2 + \cdots$.

* Received November 21, 1946.

[1] See the list of references at the end.

[2] The numbers beside the authors refer to the order of their references as given in the list.

The matrix $P = (p_{ij})$ is called the *matrix of coefficients* of the set $\{p_n(z)\}$. If P_1 and P_2 are the matrices of coefficients of the respective basic sets $\{p_n(z)\}$ and $\{q_n(z)\}$, then it has been shewn by Whittaker (4) that the matrix $P_1 \cdot P_2$ will be the matrix of coefficients of the basic set $\{u_n(z)\}$ given by

$$u_n(z) = p_{n0}q_0(z) + p_{n1}q_1(z) + p_{n2}q_2(z) + \cdots. \tag{4}$$

The set $\{u_n(z)\}$ is defined to be the *product* of the sets $\{p_n(z)\}$ and $\{q_n(z)\}$ in the given order. It was proposed by Professor Whittaker to study the convergence of the product set, or more precisely, to find out when the product set $\{u_n(z)\}$ is effective. In this note the problem is considered for the case where $\{p_n(z)\}$ is a Cannon set and $\{q_n(z)\}$ is a set for which $\overline{\lim_{n\to\infty}} D_n/n$ is finite, where D_n is defined by Whittaker (4) to be the degree of the polynomial of highest degree in (1).

Let $\{p_n(z)\}$ and $\{q_n(z)\}$ be basic sets of polynomials; then the polynomial $p_n(z)$ can be uniquely expressed as a finite linear combination of the polynomials $\{q_n(z)\}$ in the form

$$p_n(z) = \sum_i \tilde{\omega}_{ni} q_i(z). \tag{5}$$

Suppose that $f(z)$ is any function that can be represented by the basic series of $\{p_n(z)\}$ in some circle about the origin, in the form $f(z) = \sum_{n=0}^{\infty} a_n p_n(z)$. Associated with such a function there is, therefore, a series $\sum c_n q_n(z)$, where

$$c_n = a_0\tilde{\omega}_{0n} + a_1\tilde{\omega}_{1n} + a_2\tilde{\omega}_{2n} + \cdots. \tag{6}$$

This series represents $f(z)$ in $|z| \leq R$, if it converges uniformly to $f(z)$ in $|z| \leq R$. The number N_n of nonzero coefficients in (5) is assumed to be such that

$$N_n^{1/n} \to 1, \text{ as } n \to \infty. \tag{7}$$

Write $\phi_n(R) = \sum_i |\tilde{\omega}_{ni}| B_i(R)$, where $B_n(R)$ is the maximum modulus of $q_n(z)$ in $|z| \leq R$. I shall prove the following

LEMMA. *Let $\{p_n(z)\}$ be a basic set of polynomials and suppose that the basic set $\{q_n(z)\}$ satisfies (7), and that for any value of $R > 0$, $\overline{\lim_{n\to\infty}} \{\phi_n(R)/A_n(R)\}^{1/n} > 1$; then there is a function $f(z)$, which is represented by the basic series of $\{p_n(z)\}$ in $|z| \leq R$, and yet the series $\sum c_n q_n(z)$ does not represent it in $|z| \leq R$.*[3]

[3] The proof of the lemma is similar to the proof of the above theorem of Cannon (1).

For, let

$$\overline{\lim_{n\to\infty}}\,\{\phi_n(R)/A_n(R)\}^{1/n} \geq \alpha > 1. \tag{8}$$

Applying Hadamard's three circles theorem we can easily construct the sequence of numbers (R_n) such that

$$1/\alpha_1 \leq \{A_n(R)/A_n(R_n)\}^{1/n} \leq 1/\alpha_2, \qquad \text{when } 1 < \alpha_2 < \alpha_1 < \alpha. \tag{9}$$

Combining (8) and (9) we obtain

$$\overline{\lim_{n\to\infty}}\,\{\phi_n(R)/A_n(R_n)\}^{1/n} > 1. \tag{10}$$

It follows that the series $\sum |\,\bar{\omega}_{ns}\,|/A_n(R_n)$ may

(a) either diverge for at least one value of s, s_1 say, and in this case the required function will be the function $f(z) = \sum_{n=0}^{\infty} p_n(z)e^{-i\beta_{ns_1}}/A_n(R_n)$, where $\beta_{ns_1} = \arg \bar{\omega}_{ns_1}$, or

(b) converge for all values of s. Here applying (7) and (10) we construct an infinite sequence of pairs of integers (m_r, q_r) such that $m_{r+1} > m_r$, $q_{r+1} > q_r$ and

$$|\,\bar{\omega}_{m_i,q_i}\,|\,B_{qr}(R) > A_{m_r}(R_{m_r}). \tag{11}$$

Now by the convergence of the series $\sum |\,\bar{\omega}_{ns}\,|/A_n(R_n)$ and the construction of the matrix $(\bar{\omega}_{ij})$ we construct, in a successive manner, a sequence of integers (λ_k) and a subsequence $(\bar{\omega}_{b_kd_k})$ out of the sequence $(\bar{\omega}_{m_rq_r})$ such that $\bar{\omega}_{b_1d_1} = \bar{\omega}_{m_1q_1}$, $b_k > \lambda_{k-1}$, $\bar{\omega}_{nd_k} = 0$ for $n < b_k$, and $\sum_{n=\lambda_k}^{\infty} |\,\bar{\omega}_{nd_k}\,|/A_n(R_n) < \frac{1}{2}\,|\,\bar{\omega}_{b_kd_k}\,|/A_{b_k}(R_{b_k})$. The required function in this case will be the function $f(z) = \sum a_n p_n(z)$, for which $a_n = 1/A_n(R_n)$ when $n = b_1, b_2, b_3, \cdots$, and $a_n = 0$ otherwise.

We now consider the main problem and set

$$z^n = \sum_i \lambda_{ni} q_i(z). \tag{12}$$

Then from (1) and (12) and the definition of the product set it can be easily shewn that

$$z^n = \sum_i u_i(z) \sum_r \lambda_{nr}\pi_{ri}, \tag{13}$$

and [4]

[4] The relation (14) attributes an interesting meaning to the concept of product sets. Let $f(z) = \Sigma a_n z^n$ be represented in a series of polynomials of the form $\Sigma c_n p_n(z)$; then

$$q_n(z) = \sum_i \pi_{ni} u_i(z). \tag{14}$$

Write

$$\psi_n(R) = \sum_i |\pi_{ni}| M_i(R), \tag{15}$$

where $M_n(R)$ is the maximum modulus of $u_n(z)$ in $|z| \leq R$. For the sake of uniformity we shall write $\delta_n^{(1)}(R)$ for $\delta_n(R)$ and $\delta^{(1)}(R)$ for $\delta(R)$ as given by (2) and (3) and $\delta_n^{(2)}(R)$ and $\delta^{(2)}(R)$ and $\Delta_n(R)$ and $\Delta(R)$ for the corresponding expressions for the sets $\{q_n(z)\}$ and $\{u_n(z)\}$, respectively. We first prove the following

THEOREM I. *Let $\{p_n(z)\}$ be a Cannon set and let the set $\{q_n(z)\}$ be subject to the condition that $\overline{\lim_{n\to\infty}} D_n/n$ is finite and suppose that $\{q_n(z)\}$ is effective in $|z| \leq R$. Then the necessary and sufficient condition for the product set $\{u_n(z)\}$ to be effective in $|z| \leq R$ is that*

$$\overline{\lim_{n\to\infty}} \{\psi_n(R)/B_n(R)\}^{1/n} = 1. \tag{16}$$

The necessity follows from the above lemma, noting that when $\{p_n(z)\}$ is a Cannon set then the expression in (14) will satisfy (7). Also if both the sets $\{u_n(z)\}$ and $\{q_n(z)\}$ are effective in $|z| \leq R$, then the basic series of $\{u_n(z)\}$ will represent, in $|z| \leq R$, every function represented by the basic series of $\{q_n(z)\}$ in $|z| \leq R$. Applying the above lemma, (16) follows at once.

To prove the sufficiency of (16), a positive number K is chosen for the set $\{q_n(z)\}$ such that

$$D_n/n < K, \text{ for all values of } n > n_0, \text{ say.} \tag{17}$$

Now from (13) we obtain

$$\Delta_n(R) < \sum_r |\lambda_{nr}| \sum_i |\pi_{ri}| M_i(R) < \delta_n^{(2)}(R)\psi_{t_n}(R)/B_{t_n}(R),$$

where $\psi_{t_n}(R)/B_{t_n}(R) = \max_r \psi_r(R)/B_r(R) \geq 1$. If t_n remains finite then

$$\Delta(R) = \overline{\lim_{n\to\infty}} \{\Delta_n(R)\}^{1/n} \leq \delta^{(2)}(R). \tag{18}$$

If, on the other hand, t_n tends to infinity with n, then in view of (17) we have, for $n > n_0$,

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according to (14) a representation of $\Sigma a_n q_n(z)$ will be $\Sigma c_n u_n(z)$ and it is seen that the last two series are derived from the former two series by putting in each $q_n(z)$ instead of z^n. The question of product sets is therefore equivalent to that of finding the conditions governing the series $\Sigma c_n p_n(z)$ associated with the function $\Sigma a_n z^n$ in order that the series $\Sigma c_n u_n(z)$ should represent the function $\Sigma a_n q_n(z)$.

$$\{\Delta_n(R)\}^{1/n} < \{\delta_n^{(2)}(R)\}^{1/n} \cdot \{\psi_{t_n}(R)/B_{t_n}(R)\}^{k/t_n},$$

and applying (16), (18) follows also. Since the set $\{q_n(z)\}$ is effective in $|z| \leq R$, then according to (3), $\delta^{(2)}(R) = R$, which ensures that the set $\{u_n(z)\}$ is also effective in $|z| \leq R$, as required.

Now suppose that $\lim_{n\to\infty} \{B_n(R)/R^n\}^{1/n}$ exists and is finite. In fact let

$$\lim_{i\to\infty} \{B_n(R)/R^n\}^{1/n} = a < \infty. \tag{19}$$

The following theorem defines completely the region where the product set $\{u_n(z)\}$ is effective.

THEOREM II. *Let $\{p_n(z)\}$ be a Cannon set and let the set $\{q_n(z)\}$ be such that $\lim_{i\to\infty} D_n/n$ is finite, and suppose that $\{q_n(z)\}$ is effective in $|z| \leq R$, where it satisfies* (19), *and $\delta^{(1)}(aR)$ is continuous. Then the product set $\{u_n(z)\}$ will be effective in $|z| \leq R$, if, and only if, $\{p_n(z)\}$ is effective in $|z| \leq aR$.*

For let $\rho = aR$ and choose the numbers R_1, R_2, R_3 and R_4 such that

$$R_1 < R_2 < \rho < R_3 < R_4. \tag{20}$$

Then according to (19) there exists a positive integer n_0 such that

$$R_2^n < B_n(R) < R_3^n, \tag{21}$$

for $n > n_0$. Hence from (4), (20) and (21) we get

$$\begin{aligned} M_n(R) &< \{\sum_{r\leq n_0} + \sum_{r>n_0}\} |p_{nr}| B_r(R) \\ &< (n_0+1)B_s(R) \cdot A_n(R_4)/R_4^s + A_n(R_4) \sum_{r=0}^{\infty} (R_3/R_4)^r \cdot \\ &< A_n(R_4) \cdot S(R, R_3, R_4), \end{aligned} \tag{22}$$

where $B_s(R)/R_4^s = \max_{r\leq n_0} B_r(R)/R_4^r$, so that $S(R, R_3, R_4)$ is finite. In a similar manner it can be shown, by the aid of (20) and (21), that

$$A_n(R_1) < M_n(R)/T(R, R_1, R_2), \tag{23}$$

where $T(R, R_1, R_2)$ is positive and finite. Combining (22) and (23) together we get, in view of (15) and (21),

$$T(R, R_1, R_2) \cdot \delta_n^{(1)}(R_1)/R_3^n < \psi_n(R)/B_n(R) < \delta_n^{(1)}(R_4) \cdot S(R, R_3, R_4)/R_2^n.$$

Since R_1 and R_4 can be taken as near to ρ as we please and since $\delta^{(1)}(\rho)$ is continuous, then, making n tend to infinity, we conclude that

$$\overline{\lim_{i\to\infty}} \{\psi_n(R)/B_n(R)\}^{1/n} = \delta^{(1)}(\rho)/\rho. \tag{24}$$

The required result now follows by appealing to the above theorem of Cannon and to Theorem I.

The following corollaries are direct applications of Theorem II for some important special cases:

COROLLARY I. *Let* $\{p_n(z)\}$ *be a Cannon set and let the set* $\{q_n(z)\}$ *be such that* $\overline{\lim_{n\to\infty}} D_n/n$ *is finite. Suppose that the set* $\{q_n(z)\}$ *is effective in* $|z| \leq R$, *where* $\lim_{n\to\infty} \{B_n(R)/R^n\}^{1/n} = 1$, *and* $\delta^{(1)}(R)$ *is continuous. Then the product set* $\{u_n(z)\}$ *is effective in* $|z| \leq R$, *if and only if,* $\{p_n(z)\}$ *is effective in* $|z| \leq R$.[5]

COROLLARY II. *Let* $\{p_n(z)\}$ *be a Cannon set and let* $\{q_n(z)\}$ *be a simple set in which the coefficient of* z^n *in* $q_n(z)$ *is unity. Suppose that* $\{q_n(z)\}$ *is effective in* $|z| \leq R$, *where* $\delta^{(1)}(R)$ *is continuous. Then the product set* $\{u_n(z)\}$ *is effective in* $|z| \leq R$, *if, and only if,* $\{p_n(z)\}$ *is effective in* $|z| \leq R$.

The result follows from Corollary I, if we note that for such a simple set $\{q_n(z)\}$, $\lim_{t\to\infty} \{B_n(R)/R^n\}^{1/n} = 1$.

COROLLARY III. *Let* $\{p_n(z)\}$ *be a set for which* $\lim_{n\to\infty} D_n/n = 1$, *and let the set* $\{q_n(z)\}$ *be such that* $\overline{\lim_{n\to\infty}} D_n/n$ *is finite and suppose that* $\{q_n(z)\}$ *is effective in* $|z| \leq R$, *where* $\overline{\lim_{n\to\infty}} \{B_n(R)/R^n\}^{1/n} \geq 1$. *Then the product set* $\{u_n(z)\}$ *is effective in* $|z| \leq R$, *if* $\{p_n(z)\}$ *is effective in* $|z| \leq R$.

For, according to Whittaker (4), $\delta^{(1)}(R)/R < \delta^{(1)}(r)/r$, for $r < R$. Combining this with (24) the required result is obtained.

It will be observed that Theorem II gives us instances for the case in which only the 'inner' set $\{q_n(z)\}$ is effective and yet the product set is effective in the same circle. It also affords us with instances for the case where both the sets $\{p_n(z)\}$ and $\{q_n(z)\}$ are effective in $|z| \leq R$, and yet the product set is not effective in $|z| \leq R$. These instances are illustrated in the following example:

Example I. Consider the basic sets $\{p_n(z)\}$ and $\{q_n(z)\}$ given by $p_n(z) = 1 + z^n + z^{2n}/4^n$, when n is odd, and $p_n(z) = z^n$ when n is even, and $q_n(z) = z^n + 2^n z^{n+1}$ when n is odd, and $q_n(z) = 2^n z^{n-1} + z^n$ when $n > 0$ is even and $q_0(z) = 1$.

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[5] The condition that $\lim \{B_n(R)/R^n\}^{1/n} = 1$ has been obtained as a sufficient condition for the effectiveness of $\{u_n(z)\}$ if both $\{p_n(z)\}$ and $\{q_n(z)\}$ are effective.

It is easily seen that $\{p_n(z)\}$ is effective in $|z| \leq R$ for the values of R for which $1 \leq R \leq 4$, and that $\{q_n(z)\}$ is effective in the whole plane. Also it is noted that $\lim_{n\to\infty} \{B_n(R)/R^n\}^{1/n} = 2$. Forming the product set $\{u_n(z)\}$ of the sets $\{p_n(z)\}$ and $\{q_n(z)\}$ we get

$$u_n(z) = 1 + z^n + 2^n z^{n+1} + z^{2n-1} + z^{2n}/4^n, \text{ when } n \text{ is odd, and}$$

$$u_n(z) = 2^n z^{n-1} + z^n, \text{ when } n > 0 \text{ is even, and } u_0(z) = 1.$$

It can be easily shown that $\overline{\lim_{n\to\infty}} \{\psi_n(R)/B_n(R)\}^{1/n} = 1$ when $\frac{1}{2} \leq R \leq 2$ and thus, by Theorem I, the set $\{u_n(z)\}$ is effective in $|z| \leq R$ for the values of R for which $\frac{1}{2} \leq R \leq 2$, as is expected from Theorem II. Also we note that in the interval $\frac{1}{2} \leq R < 1$, the set $\{q_n(z)\}$ is effective in $|z| \leq R$, while the set $\{p_n(z)\}$ is not effective and yet the product set $\{u_n(z)\}$ is effective. Also in the interval $2 < R \leq 4$, both the sets $\{p_n(z)\}$ and $\{q_n(z)\}$ are effective in $|z| \leq R$, and yet the product set $\{u_n(z)\}$ is not effective.

Now the condition that $\lim_{n\to\infty} \{B_n(R)/R^n\}^{1/n}$ exists and is finite, is necessary for the truth of Theorem II in the sense that if this condition is not satisfied then the theorem is no longer true. In the first place if $\lim_{n\to\infty} \{B_n(R)/R^n\}^{1/n}$ is infinite, then instances can be found in which both the sets $\{p_n(z)\}$ and $\{q_n(z)\}$ are effective in the whole plane while the product set $\{u_n(z)\}$ is nowhere effective.[6]

Moreover, if

$$(25) \qquad 0 < a = \underline{\lim_{n\to\infty}} \{B_n(R)/R^n\}^{1/n} < \overline{\lim_{n\to\infty}} \{B_n(R)/R^n\}^{1/n} = b \leq \infty,$$

then Theorem II no longer holds, as is seen from the following

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[6] Thus consider, for example, the basic sets $\{p_n(z)\}$ and $\{q_n(z)\}$ given by

$p_n(z) = z^n + z^{2n}/n^n$, when n is odd, and $p_n(z) = z^n$, when n is even, and

$q_n(z) = z^n + n^n z^{n+1}$, when n is odd, and $q_n(z) = (n-1)^{2(n-1)} z^{n-1} + z^n$ when $n > 0$ is even, and $q_0(z) = 1$.

Accordingly the product set $\{u_n(z)\}$ will be given by

$u_0(z) = 1$, and $u_n(z) = (n-1)^{2(n-1)} z^{n-1} + z^n$, when $n > 0$ is even, and

$u_n(z) = z^n + n^n z^{n+1} + (2n-1)^{2(2n-1)} z^{2n-1}/n^n + z^{2n}/n^n$, when n is odd.

It is easily seen that $\{B_n(R)/R^n\}^{1/n}$ tends to infinity with n, that both $\{p_n(z)\}$ and $\{q_n(z)\}$ are effective in the whole plane, and finally that $\overline{\lim_{n\to\infty}} \{\psi_n(R)/B_n(R)\}^{1/n} = \infty$ for all values of $R > 0$.

THEOREM III. *Let* $\{q_n(z)\}$ *be a basic set for which* $\overline{\lim_{n\to\infty}} D_n/n$ *is finite and suppose that* $\{q_n(z)\}$ *is effective in* $|z| \leq R$, *where it satisfies* (25). *Then there exists a Cannon set* $\{p_n(z)\}$ *which is effective in* $|z| \leq r$, *for all positive values of* $r \leq bR$, *such that the product set* $\{u_n(z)\}$ *is not effective in* $|z| \leq R$.

For, write $aR = \rho$, then, given any number $R_1 > \rho$, there exists an infinite sequence of positive integers (l_r) such that

$$B_{l_r}(R) < R_1^{l_r} \qquad (r = 1, 2, 3, \cdots). \tag{26}$$

Writing $\{B_n(R)/R^n\}^{1/n} = b_n$, then there exists another finite sequence of positive integers (m_r), having no common members with (l_r) such that

$$b_{m_r} \to b, \text{ as } r \to \infty. \tag{27}$$

We form the infinite sequence of triads of integers $(\lambda_i, \mu_i, \nu_i)$ for $i = 1, 2, 3, \cdots$, where

$$\begin{cases} (\lambda_i) \text{ belongs to } (l_r) \text{ and } (\mu_i) \text{ belongs to } (m_r), \text{ and} \\ \lambda_i < \mu_i \text{ and } \nu_i = \mu_i - \lambda_i. \end{cases} \tag{28}$$

The required Cannon set $\{p_n(z)\}$ is given by

$$p_{\lambda_i}(z) = z^{\lambda_i} + (z^{\mu_i}/(b_{\mu_i} \cdot R)^{\nu_i}) \text{ and } p_n(z) = z^n, \text{ otherwise.} \tag{29}$$

It can be easily seen, using (27), (28) and (29), that the set $\{p_n(z)\}$ is either effective in the whole plane if $\lim_{i\to\infty} (\mu_i/\lambda_i) = 1$, or else is effective in $|z| \leq r$, for all positive values of $r \leq bR$. Now the product set $\{u_n(z)\}$ is given by

$$u_{\lambda_i}(z) = q_{\lambda_i}(z) + q_{\mu_i}(z)/(b_{\mu_i} \cdot R)^{\nu_i}, \text{ and } u_n(z) = z^n, \text{ otherwise.}$$

Hence in view of (26) and (28) we have

$$\frac{\psi_{\lambda_i}(R)}{B_{\lambda_i}(R)} = \frac{M_{\lambda_i}(R) + (1/(b_{\mu_i} \cdot R)^{\nu_i}) M_{\mu_i}(R)}{B_{\lambda_i}(R)} > \left(\frac{b_{\mu_i} \cdot R}{R_1}\right)^{\lambda_i}.$$

Allowing i to tend to infinity we conclude, according to (27), that

$$\overline{\lim_{n\to\infty}} \{\psi_n(R)/B_n(R)\}^{1/n} \geq \lim_{i\to\infty} \{\psi_{\lambda_i}(R)/B_{\lambda_i}(R)\}^{1/\lambda_i} \geq bR/R_1 > 1,$$

since we can take R_1 as near to ρ as we please. Hence by Theorem I the set $\{u_n(z)\}$ is not effective in $|z| \leq R$, as required.

I append the following illustrative example:

Example II. Consider the basic set $\{q_n(z)\}$ given by

$$q_n(z) = z^n + 2^n z^{n+2} \quad \text{when } n = 3h, \text{ and}$$
$$q_n(z) = 2^n z^{n-1} + z^n + 2^n z^{n+1} \quad \text{when } n = 3h+1, \text{ and}$$
$$q_n(z) = z^n \quad \text{when } n = 3h+2,$$

for $h = 0, 1, 2, \cdots$. It can be easily shewn that $\{q_n(z)\}$ is effective in the whole plane, and

$$1 = \underline{\lim}_{n\to\infty} \{B_n(R)/R^n\}^{1/n} < \overline{\lim}_{n\to\infty} \{B_n(R)/R^n\}^{1/n} = 2.$$

Forming the product set $\{u_n(z)\}$ of the sets $\{p_n(z)\}$ as given in Example I and $\{q_n(z)\}$ as given above, we get

$$u_n(z) = z^n + 2^n z^{n+1}; \quad \text{for } n = 6h,$$
$$u_n(z) = 1 + z + 2^n z^{n-1} + z^n + 2^n z^{n+1} + z^{2n}/4^n; \quad \text{for } n = 6h+1,$$
$$u_n(z) = z^n; \quad \text{for } n = 6h+2,$$
$$u_n(z) = 1 + z + z^n + 2^n z^{n+1} + z^{2n}/4^n + z^{2n+1}; \quad \text{for } n = 6h+3,$$
$$u_n(z) = 2^n z^{n-1} + z^n + 2^n z^{n+1}; \quad \text{for } n = 6h+4,$$
$$u_n(z) = 1 + z + z^n + z^{2n-1} + z^{2n}/4^n + {}^{2n+1}; \quad \text{for } n = 6h+5,$$

$h = 0, 1, 2, \cdots$. It is easily seen that $\overline{\lim}_{n\to\infty} \{\psi_n(R)/B_n(R)\}^{1/n} = 1$ only when $R = 1$ and is otherwise greater that 1, so that the set $\{u_n(z)\}$ is only effective in $|z| \leq 1$.

FAROUK I UNIVERSITY,
ALEXANDRIA, EGYPT.

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ELIMINANTS.*

By JOSEPH MILLER THOMAS.

From the theory of functional dependence is known the existence of polynomials $F(x, y)$ which vanish identically in t when the indeterminates x, y are replaced by given polynomials in the single indeterminate t. A specific polynomial $E(x, y)$ of this sort is defined in the present paper (1) as the eliminant of two given polynomials. This polynomial has been employed by Perron [1; p. 223]. Taking the complex numbers for coefficient field (the extension to fields of characteristic zero is immediate), here we develop certain properties of this eliminant. Firstly, the eliminant is the k-th power of an irreducible polynomial f. Secondly, the eliminant is reducible (that is, $1 < k$) if and only if the two polynomials $x(t)$, $y(t)$ are also polynomials in a second indeterminate u which itself is a polynomial of degree k in the original indeterminate t. Thirdly, whereas elimination of t by the eliminant gives Af^k, where A is a non-zero complex number, elimination by the usual division process gives hf, where h is an extraneous factor, whose nature is precisely stated, namely, h is the product of non-negative integral powers of the initials appearing in the division sequence and h may involve x and y. Fourthly, the eliminant is used to give algebraic conditions that a single polynomial $y(t)$ be a polynomial in a second indeterminate u whose degree satisfies $1 < \deg_t u < \deg_t y$. An elegant solution of the last problem (for a single polynomial and a parameter of any degree) from group-theoretic and function-theoretic considerations has previously been given by Ritt [2].

1. The eliminant. The parametric equations

$$x = a_0 t^m + \cdots + a_m, \qquad y = b_0 t^n + \cdots + b_n, \qquad a_0 b_0 \neq 0 \tag{1.1}$$

define a curve which is the locus of the point (x, y). Let the resultant of the two polynomials be denoted by $R(a_m, b_n)$, a notation which puts in evidence its dependence on the constant terms of the polynomials. Put

$$E(x, y) = R(a_m - x, b_n - y). \tag{1.2}$$

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The equation

$$E(x, y) = 0 \tag{1.3}$$

is a necessary and sufficient condition that for a given (x, y) there exist a t satisfying (1.1). Hence (1.3) is the equation of the curve. Its left member will be called the *eliminant* of the two polynomials. If the values (1.1) are introduced into E, then

$$E(x, y) \equiv 0 \tag{1.4}$$

in t. The eliminant satisfies

$$\deg_{x,y} E = \max(m, n), \qquad \deg_x E = n, \qquad \deg_y E = m; \tag{1.5}$$

all of these relations are independent of the values given (x, y).

2. Reducibility. Let the factorization of E into irreducible factors be

$$E(x, y) = f(x, y)g(x, y) \cdots.$$

Under the substitution (1.1) at least one of the factors, say $f(x, y)$, becomes identically zero. If (x, y) is a root of $g(x, y)$, there exists a t satisfying (1.1). Hence (x, y) is also a root of $f(x, y)$, f and g are associates and

$$E(x, y) = Af^k, \tag{2.1}$$

where A is a non-zero complex number, f is an irreducible polynomial in x, y and k is a positive integer. The curve is got by imagining the points of the unicursal curve $f(x, y) = 0$ traced k times. From (2.1) we have

$$m = k \deg_y f, \qquad n = k \deg_x f. \tag{2.2}$$

Suppose that x, y are also polynomials in a parameter u, which itself is a polynomial in t:

$$u = c_0 t^p + c_1 t^{p-1} + \cdots + c_p, \qquad c_0 \neq 0. \tag{2.3}$$

Denote by $E(x, y; u)$ the E formed with respect to u, and by $E(x, y; t)$ that previously considered. Since the point set constituting the solution of $E(x, y) = 0$ is the same for both parameters, f can be taken the same for u as for t. Hence $E(x, y; u) = Bf^l$, and

$$E(x, y; t) = C[E(x, y; u)]^p, \qquad k = pl. \tag{2.4}$$

The parameter t will be called *reducible* or *irreducible* according as there does or does not exist for $1 < p$ a polynomial u in terms of which x and y can be simultaneously expressed. From (2.4) it follows that if the parameter t is reducible, so also is the eliminant $E(x, y; t)$. The converse of this proposition is also true as will be seen in **4.**

3. Highest common factor. Generalizing (1.2) put

$$E_i(x, y) = R_i(a_m - x, b_n - y),$$

where R_i is the i-th subresultant and $E_0 = E$. Let a positive integer l be defined by

$$(3.1) \qquad E_0(x, y) \equiv 0, \cdots, E_{l-1}(x, y) \equiv 0, E_l(x, y) \not\equiv 0,$$

the sign $\equiv$ being read "equals identically in t." The curves $E_0 = 0$, $E_k = 0$ intersect in a finite number of points, which are singular on $f = 0$. For an ordinary (x, y) the highest common factor of

$$(3.2) \qquad r_0 = a_0 t^m + \cdots + a_m - x, \qquad r_1 = b_0 t^n + \cdots + b_n - y$$

has degree l and is given by

$$(3.3) \qquad E_l(x, y) t^l + \cdots,$$

where the unwritten terms are easily specified polynomials in (x, y). The locus of points at which the discriminant of (3.3) vanishes intersects $E(x, y) = 0$ in a finite number of points. Except at these and singular points, (3.3) has exactly l distinct roots $t_1, \cdots, t_l$ all of which give the same point (x, y) on $f = 0$. That the roots of (3.3) may not all be distinct at an ordinary point is shown by $x = t^2$, $y = t^2$, $E(x, y) = -(x - y)^2$.

It is now convenient to suppose the notation chosen so that $n \leqq m$ and consequently $\deg_{x,y} E = m$. By avoiding a finite number of points, it is possible to choose a point (x, y) such that $E_l(x, y) \neq 0$ and such that the line $X = x$ meets the curve $f = 0$ in m/k distinct points. The m parameter values of these points fall into m/k sets of l each. Hence $k = l$ and the positive integer k in (2.1) can be determined by

$$(3.4) \qquad E_0(x, y) \equiv 0, \cdots, E_{k-1}(x, y) \equiv 0, E_k(x, y) \not\equiv 0.$$

4. The fundamental theorem. If t and s are indeterminates, the polynomial $x(t) - x(s)$ is divisible by $t - s$. If it is written as the product of two factors, the factorization can be given the form

$$x(t) - x(s) = (t^k + \cdots - s^k)(a_0 t^{m-k} + \cdots + a_0 s^{m-k}), \tag{4.1}$$

the first factor being divisible by $t - s$. The highest common factor of $x(t) - x(s)$ and $y(t) - y(s)$ can therefore be assumed in the form

$$u(t, s) = t^k + \cdots - s^k.$$

Let the polynomials $u(t, s_1)$ and $u(t_1, s)$ have the roots $s_1, \cdots, s_k$ and $t_1, \cdots, t_k$ respectively. The parameter values $s_1, \cdots, s_k$ all give the same point, say (x, y) and the $t_1, \cdots, t_k$ the same point, say (x_0, y_0). Write

$$u(t, s) = t^k + \alpha_1(s) t^{k-1} + \cdots + \alpha_k(s), \tag{4.2}$$

where $\alpha_1, \cdots, \alpha_{k-1}$ are polynomials of degree at most $k - 1$ and α_k is of degree k. Consider an ordinary point s_1. Since (4.2) is an associate of (3.3) its coefficients depend only on (x, y) and

$$\alpha_j(s_1) = \cdots = \alpha_j(s_k) \qquad (j = 1, 2, \cdots, k).$$

The equations

$$\alpha_j(s) - \alpha_j(s_1) = 0 \qquad (j = 1, 2, \cdots, k - 1)$$

accordingly are identities. Moreover the polynomial $\alpha_k(s) - \alpha_k(s_1)$ has the roots $s_1, \cdots, s_k$ and therefore is $-u(s, s_1)$. Writing $u(t)$ for $u(t, s_1)$ we therefore have

$$u(t, s) = u(t) - u(s). \tag{4.3}$$

Let $u(t)$ be fixed by assigning an s_1 corresponding to an ordinary point. The k parameter values belonging to each point (x, y) on $E(x, y) = 0$ also belong to some point (x, y) on the curve

$$x = x(t), \qquad y = u(t). \tag{4.4}$$

The eliminant of (4.4) is therefore the k-th power of an irreducible polynomial of degree 1 in x. Hence x is a polynomial in u. A similar argument applies to y and we have

THEOREM 4.1. *The eliminant is reducible if and only if the parameter is reducible.*

The reducibility of the eliminant is equivalent to the reducibility of the resultant in which the two coefficients a_m, b_n (and only those) are indeterminates.

Once it has been established that there are k parameter values giving each point, Lüroth's theorem shows that x, y are rational functions (not polynomials) in a reduced parameter and that the reduced parameter is a rational function of x, y. That the second result cannot be strengthened in the present case is shown by the example:

$$x = t^6, \quad y = t^4, \quad E(x,y) = (x^2 - y^3)^2, \quad u(t) = t^2 - s_1^2 = (x/y) - (x_0/y_0).$$

If u were a polynomial in (x, y), we should have identically

$$x - yg(x,y) = h(x,y)(x^2 - y^3),$$

where g, h are polynomials. Evaluation for $y = 0$ gives the contradiction $x = x^2 h(x, 0)$.

5. Connection with division process. Still under the assumption $n \leqq m$, let r_2 be the remainder when $b_0^{m-n+1} r_0$ is divided by r_1 for indeterminate x, y (see (3.2) for the definition of r_0, r_1). In this way form a division sequence

$$r_0, r_1, r_2, \cdots, r \tag{5.1}$$

whose last term r is not zero and does not contain t. The polynomials in (5.1) are equal to the corresponding polynomials in the division sequence formed for the irreducible parameter, the only possible change being in the parameter. The degree differences for the original parameter will be k times those for the irreducible parameter.

By reference to [3; § 6] it is seen that $E_j(x, y)$ vanishes identically in x, y if j is not divisible by k. This can be used to modify (3.4).

It can be shown [3; (4.8)] that

$$\pm g(x,y) R_0(a_m - x, b_n - y) = I^d, \tag{5.2}$$

where $g(x, y)$ is the product of non-negative integral powers of initials of polynomials which precede $r(x, y)$ in the sequence (5.1), where I is the initial of $r(x, y)$ and where d is the last degree difference. Since in this case r does not involve t, the initial of $r(x, y)$ is $r(x, y)$ itself. Hence (5.2) becomes

$$\pm Ag(x,y)[f(x,y)]^k = [r(x,y)]^d. \tag{5.3}$$

It is clear from this relation that f divides r and that d (which is a multiple

of k) can exceed k only if f divides an initial of a polynomial preceding r. Hence we have

$$h(x,y)f(x,y) = r(x,y), \tag{5.4}$$

where f is the irreducible factor of the eliminant and h is a non-zero constant times the product of non-negative integral powers of initials of polynomials preceding r in the division sequence.

An important consequence is that *eliminating by the division process may introduce an extraneous factor* $h(x,y)$, *whereas eliminating by the eliminant never does.* An example follows.

$$x = t^5 - t + 2, \qquad y = t^3 + t^2 + 2,$$

$$E(x,y) = f(x,y) = 116 - 54x - 186y + 4x^2 + 67xy + 126y^2 + x^3$$
$$- 5x^2y - 27xy^2 - 48y^3 + 5xy^3 + 10y^4 - y^5,$$

$$h(x,y) = -(3-y)^2.$$

In the reducible case, the division process may yield the irreducible factor of $E(x,y)$, for example,

$$x = t^2 + t + 1, \qquad y = t^2 + t + 2, \qquad -x + y - 1 = r.$$

6. Polynomials in an unknown polynomial. Let us consider the reducibility of the parameter in the case of a single polynomial x. Since we may always write $x = u$, where u is of degree m, the parameter is always reducible, if the definition for reducibility is carried over directly from the case of two polynomials. Accordingly, we make the following definition: the parameter t in $x(t)$ is reducible if and only if x can be expressed as a polynomial in u, where u is a polynomial in t and $1 < \deg_t u < \deg_t x$. A parameter reducible for a pair of polynomials is then reducible for a polynomial x of the pair if and only if the irreducible factor of the eliminant is not linear in the other polynomial y of the pair. A single polynomial with parameter reducible in the sense just given is what Ritt [2] has called a *composite* polynomial.

In (1.1) let y be the polynomial with given complex coefficients. Let m be a fixed divisor of n satisfying $1 < m < n$. Let the coefficients of x be unknowns. The conditions that $E(x,y)$ be a perfect n/m power together with the inequation $a_0 \neq 0$ constitute an algebraic system which we shall denote by $S_m(y)$.

THEOREM 6.1. *The parameter in y is reducible if and only if $S_m(y)$ is consistent for some m satisfying*

$$m \mid \deg_t y, \qquad 1 < m < \deg_t y.$$

Thus by discussing certain systems of algebraic equations it can be decided whether a reduced parameter exists and any reduced parameter can be determined.

DUKE UNIVERSITY

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TAUBER'S THEOREM AND ABSOLUTE CONSTANTS.*

By PHILIP HARTMAN.

Let $a_1, a_2, \cdots$ be a sequence of real numbers. The theorem of Tauber [2] which states that an Abel summable series Σa_n is convergent when

$$a_1 + 2a_2 + \cdots + na_n = o(n), \qquad n \to \infty,$$

has recently been refined by Wintner [3] to a theorem involving an absolute constant as follows: There exists an absolute constant τ having the property that

$$(1) \quad \limsup_{r\to 1-0} \Big| \sum_{m=1}^{\infty} a_m r^m - \sum_{m=1}^{-1/\log r} a_m \Big| \leqq \tau \limsup_{n\to\infty} | a_1 + 2a_2 + \cdots + na_n \rceil / n$$

for all sequences $a_1, a_2, \cdots$. This implies the existence of another absolute constant τ^* such that

$$(2) \qquad \limsup_{r\to 1-0} \Big| \sum_{m=1}^{\infty} a_m r^m - \sum_{m=1}^{-1/\log r} a_m \Big| \leqq \tau^* \limsup_{n\to\infty} | na_n |$$

for all sequences $a_1, a_2, \cdots$. The existence of this absolute constant implies the weaker theorem of Tauber that Abel summability and

$$na_n = o(1), \qquad n \to \infty,$$

assure convergence. Hadwiger [1] had refined this last quoted theorem of Tauber in terms of an absolute constant ρ, which he defines in a somewhat different manner. However, it is clear from his proof for the existence of ρ that $\rho = \tau^*$. Hadwiger has obtained the following estimates for the best value of $\rho = \tau^*$,

$$0.4858 \cdots \leqq \tau^* \leqq C + 2B = 1.01598 \cdots,$$

where $C = 0.57721 \cdots$ is the Euler-Mascheroni constant and

$$B = \int_1^{\infty} u^{-1} e^{-u} du = 0.21938 \cdots,$$

so that

$$C + B = \int_0^1 u^{-1}(1 - e^{-u})\, du = \sum_{n=1}^{\infty} (-1)^{n-1}/n \cdot n! \,.$$

Wintner has shown that

* Received November 23, 1946.

$$0.57721\cdots = C \leqq \tau^* \leqq \tau$$

and his proof for the existence of τ gives the following estimate for the best value of τ:

$$1 \leqq \tau \leqq 3 + B = 3.21938\cdots.$$

The object of this paper is to obtain the best values for τ^* and τ. In particular, it will be shown that Hadwiger's upper bound for the best value of τ^* is actually the best value; so that (2) holds with

$$\tau^* = C + 2B = 1.01598\cdots \tag{3}$$

for *all* sequences $a_1, a_2, \cdots$ and the sign of equality holds for *some* sequences. On the other hand, the best value for τ is

$$\tau = C + 2B + 2/e = 1.75174\cdots; \tag{4}$$

so that (4) satisfies (1) for *all* sequences $a_1, a_2, \cdots$ and the sign of equality holds for *some* sequences.

The proofs of (3) and (4) will be based on several simple lemmas dealing with the geometric series and the power series for $-\log(1-r)$. In what follows, k, h, r are functions of n and all of the symbols $o(1)$ refer to $n \to \infty$.

LEMMA 1. *If $k_1, k_2, \cdots$ is an increasing sequence of integers such that*

$$k = k_n = o(n), \tag{5}$$

and if r is a function of n satisfying

$$0 < r < 1 \text{ and } -n \log r \to 1, \qquad n \to \infty, \tag{6}$$

then, as $n \to \infty$,

$$\sum_{m=1}^{k} (1 - r^m)/m = o(1), \tag{7}$$

and

$$(1-r) \sum_{m=1}^{k} r^m = o(1). \tag{8}$$

To prove (7), use will be made of the inequality

$$1 - r^m < m(1-r), \qquad (0 < r < 1),$$

a consequence of the mean value theorem of differential calculus. This inequality implies that the sum on the left of (7) is majorized by

$$\sum_{m=1}^{k} (1-r) = k(1-r),$$

which is $o(1)$ in virtue of (5), since (6) is equivalent to

$$1 - r = 1/n + o(1/n). \tag{9}$$

To prove (8), it is sufficient to note that

$$(1-r)\sum_{m=0}^{k} r^m = 1 - r^{k+1} < (k+1)(1-r).$$

Hence (8) follows from (9) and (5).

LEMMA 2. *If r is a function of n satisfying* (6), *then*

$$\sum_{m=1}^{n} (1-r^m)/m = C + B + o(1), \tag{10}$$

and

$$(1-r)\sum_{m=1}^{n} r^m = 1 - 1/e + o(1). \tag{11}$$

The sum on the left of (11) is, up to a factor r,

$$1 - r^n = 1 - (1 - 1/n + o(1/n))^n$$

by (9). Hence (11) follows at once.

The sum on the left of (10) can be written as

$$\sum_{m=1}^{n} 1/m - \sum_{m=1}^{\infty} r^m/m + \sum_{m=n+1}^{\infty} r^m/m.$$

Since the first term of this expression is $\log n + C + o(1)$, and the second is $\log(1-r) = -\log n + o(1)$, it is sufficient to prove the following:

LEMMA 3. *If r is a function of n satisfying* (6), *then, as $n \to \infty$,*

$$\sum_{m=n+1}^{\infty} r^m/m = B + o(1), \tag{12}$$

and

$$(1-r)\sum_{m=n+1}^{\infty} r^m = 1/e + o(1). \tag{13}$$

First, (13) is an immediate consequence of (11) since

$$(1-r)\sum_{m=0}^{\infty} r^m = 1.$$

Next, the sum on the left of (12) can be written as the integral

$$\int_0^r t^n (1-t)^{-1} dt = \int_{-n\log r}^{\infty} n^{-1} (e^{u/n} - 1)^{-1} e^{-u} du,$$

where the last integral is obtained by an obvious change of variable. Since $n(e^{u/n}-1)\to u$, $n\to\infty$, uniformly on any bounded u-interval, since the integrand of the last integral is majorized by e^{-u}/u for all values of $u>0$ and, finally, since (6) holds, the relation (12) is a consequence of the last formula and the definition of B.

LEMMA 4. *If $h_1, h_2, \cdots$ is a sequence of increasing positive integers such that*

$$n = o(h), \text{ where } h = h_n, \tag{14}$$

and if r is a function of n satisfying (6), then, as $n\to\infty$,

$$\sum_{m=h}^{\infty} r^m/m = o(1), \tag{15}$$

and

$$(1-r)\sum_{m=h}^{\infty} r^m = o(1). \tag{16}$$

The sum occurring in (15) can be written as an integral and appraised as follows

$$\int_0^r t^h(1-t)^{-1}dt < \int_{-h\log r}^{\infty} u^{-1}e^{-u}du.$$

Since (6) and (14) imply that

$$-h\log r\to\infty, \qquad n\to\infty, \tag{17}$$

(15) follows. As to the statement (16), it is sufficient to observe that the sum on the left of (16) is $r^h = o(1)$ by (17).

The proof of (3) will now be given. It will first be shown that

$$r^* \geqq C + 2B. \tag{18}$$

The difference

$$\sum_{m=1}^{\infty} a_m r^m - \sum_{m=1}^{n} a_m \tag{19}$$

can be written in the form

$$\sum_{m=1}^{k} ma_m(r^m-1)/m + \sum_{m=k+1}^{n} ma_m(r^m-1)/m + \sum_{m=n+1}^{h} ma_m r^m/m + \sum_{m=h+1}^{\infty} ma_m r^m/m,$$

where k and h are integers such that $1<k<n<h$. Let $0=k_0, k_1, k_2, \cdots$ be an increasing sequence of integers such that

$$k_{n-1} = o(k_n), \qquad n\to\infty, \tag{20}$$

and let the sequence of numbers $a_1, a_2, \cdots$ be defined by placing

$$ma_m = (-1)^j \text{ if } k_j < m \leqq k_{j+1} \qquad (j = 0, 1, \cdots). \tag{21}$$

The corresponding difference (19), with $n = k_j$ and r a function of n satisfying (6), can be written in the form following (19), with $k = k_{j-1}$ and $h = k_{j+1}$. Then (21), (20) and Lemma 1 imply that

$$\sum_{m=1}^{k} ma_m(r^m - 1)/m = o(1); \tag{22}$$

while (21) and Lemma 1 give

$$\sum_{m=k+1}^{n} ma_m(r^m - 1)/m = (-1)^j \sum_{m=k+1}^{n} (1 - r^m)/r^m$$
$$= (-1)^j \sum_{m=1}^{n} (1 - r^m)/m + o(1).$$

Hence, by Lemma 2,

$$\sum_{m=k+1}^{n} ma_m(r^m - 1)/m = (-1)^j(C + B) + o(1). \tag{23}$$

Similarly, Lemma 4 implies that

$$\sum_{m=h+1}^{\infty} ma_m r^m/m = o(1), \tag{24}$$

and that

$$\sum_{m=n+1}^{h} ma_m r^m/m = (-1)^j \sum_{j=n+1}^{\infty} r^m/m + o(1).$$

Hence, by Lemma 3,

$$\sum_{m=n+1}^{h} ma_m r^m/m = (-1)^j B + o(1). \tag{25}$$

The relations (22), (23), (24) and (25) show that if $n = k_j$ and if r is a function of n satisfying (6), then the difference (19) is

$$(-1)^j(C + 2B) + o(1), \qquad n \to \infty.$$

Hence, (18) follows from (2) and (21).

Accordingly, to complete the proof of (3), it is sufficient to show that

$$\tau^* \leqq C + 2B.$$

But this inequality follows from Hadwiger's upper estimate for τ^*.

In order to verify (4), first

$$\tau \leqq C + 2B + 2/e \tag{26}$$

will be proved.

Let $a_1, a_2, \cdots$ be an arbitrary sequence of real numbers, and put

$$t_n = \sum_{m=1}^{n} m a_m/n, \qquad n = 1, 2, \cdots. \tag{27}$$

It can be supposed that

$$L = \limsup_{n\to\infty} | t_n | < \infty, \tag{28}$$

since otherwise (1) is trivial. Let

$$L_n = \operatorname{fin\,sup}_{m>n} | t_m | ; \tag{29}$$

so that

$$L_n \to L, \qquad n \to \infty. \tag{30}$$

By Abel's identity,

$$\sum_{m=1}^{\infty} a_m r^m = \sum_{m=1}^{\infty} m a_m r^m/m = \sum_{m=1}^{\infty} m t_m (r^m/m - r^{m+1}/(m+1)).$$

This can be simplified to

$$\sum_{m=1}^{\infty} a_m r^m = \sum_{m=1}^{\infty} t_m (r^m(1-r) + r^{m+1}/(m+1)).$$

Also, by (27) and Abel's identity,

$$\sum_{m=1}^{n} a_m = \sum_{m=1}^{n} m a_m/m = \sum_{m=1}^{n} t_m/(m+1) + n t_n/(n+1).$$

Hence, the difference (19) can be written as the sum of the following five expressions:

$$\sum_{m=1}^{k} t_m (r^m(1-r) + (r^{m+1}-1)/(m+1)), \tag{31$_1$}$$

$$\sum_{m=k+1}^{n} t_m (r^m(1-r) + (r^{m+1}-1)/(m+1)), \tag{31$_2$}$$

$$(1-r) \sum_{m=n+1}^{\infty} t_m r^m, \tag{31$_3$}$$

$$\sum_{m=n+1}^{\infty} t_m r^{m+1}/(m+1), \tag{31$_4$}$$

$$-n t_n/(n+1), \tag{31$_5$}$$

where k is an integer such that $1 < k < n$.

Let $k = k_n$ be a function of n satisfying (5) and

$$k_n \to \infty, \qquad n \to \infty, \tag{32}$$

and let r be a function of n satisfying (6). Then (28) and Lemma 1 imply that the sum (31_1) is $o(1)$ as $n \to \infty$.

To treat the sum in (31_2), it is important to notice that, since $0 < r < 1$,

$$|r^m(1-r) + (r^{m+1}-1)/(m+1)| = (1-r^{m+1})/(m+1) - r^m(1-r),$$

in view of

$$r^m(1-r) + (r^{m+1}-1)/(m+1) = (1-r)\left((m+1)r^m - \sum_{j=0}^{m} r^j\right)/(m+1) < 0.$$

Hence, the sum (31_2) is majorized by

$$L_k\left(\sum_{m=k+1}^{n}(1-r^m)/m - (1-r)\sum_{m=k+1}^{n} r^m\right).$$

In virtue of (5), (6) and Lemma 1, the factor of L_k is

$$\sum_{m=1}^{n}(1-r^m)/m - (1-r)\sum_{m=1}^{n} r^m + o(1),$$

which, by Lemma 2, is

$$C + B - 1 + 1/e + o(1). \tag{33_2}$$

In virtue of (29) and Lemma 3, the absolute values of the sums (31_3) and (31_4) can be appraised by

$$L_n(1-r)\sum_{m=n+1}^{\infty} r^m = L_n(1/e + o(1)) \tag{33_3}$$

and

$$L_n \sum_{m=n+1}^{\infty} r^m/m = L_n(B + o(1)), \tag{33_4}$$

respectively.

Finally, the absolute value of (31_5) does not exceed

$$L_n(1 + o(1)). \tag{33_5}$$

Collecting the results (33_2)-(33_5), it is seen that the difference (19) is majorized by

$$L_k(C + B - 1 + 1/e) + L_n(1/e + B + 1) + o(1).$$

In view of (29), (30) and (32), it follows that (1) holds for all sequences $a_1, a_2, \cdots$ when τ is defined by (4). This proves (26).

In order to complete the proof of (4), it remains to be shown that

$$\tau \geqq C + 2B + 2/e. \tag{34}$$

To this end let $0 = k_0, k_1, k_2, \cdots$ be an increasing sequence of integers satisfying (20). Define a sequence $a_1, a_2, \cdots$ in terms of the numbers $t_1, t_2, \cdots$ by placing

$$t_m = (-1)^j \text{ if } k_j < m \leqq k_{j+1} \qquad (j = 0, 1, 2, \cdots). \tag{35}$$

The detailed computations may be omitted. In fact, it is clear from the above construction that the difference (19), with $n = k_j$ and with r as a function of n satisfying (6), is

$$(-1)^j(C + 2B + 2/e) + o(1).$$

This implies (34) in virtue of (1) and (35).

The proof for the existence of τ given by Wintner treated the Laplace transform analogue of (1) rather than (1) itself. Actually, the above considerations are easily transcribed to this more general situation. Let $\alpha(x)$ be defined for all $x \geqq 1$ and of bounded variation on any bounded x-interval. Define $\beta(x)$ by

$$\beta(x) = \int_1^x t d\alpha(t) \text{ if } x \geqq 1.$$

Then

$$\limsup_{s \to +0} \left| \int_1^\infty e^{-st} d\alpha(t) - \int_1^{1/s} d\alpha(t) \right| \leqq \tau \limsup_{x \to \infty} |\beta(x)/x|$$

holds if τ is the number (4).

THE JOHNS HOPKINS UNIVERSITY.

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PROJECTIVE THEORY OF SURFACES AND CONJUGATE NETS IN FOUR-DIMENSIONAL SPACE.*

By CHUAN-CHIH HSIUNG.

Introduction. It is known that every non-developable surface immersed in a linear space S_4 of four dimensions has on it either a conjugate net, or else a unique one-parameter family of asymptotic curves. The purpose of this paper is to establish a purely geometric theory of the projective differential geometry of the surfaces sustaining a conjugate net.[1]

In **1** a completely integrable system of linear homogeneous partial differential equations, together with its canonical form by a geometric determination, is introduced which defines a conjugate net in space S_4 except for a projective transformation.

In **2** we study the effect on the differential equations of **1** of a group of transformations which leave invariant the parametric conjugate net N_c on an integral surface S of these equations. Some invariants and covariants of this system under the group of transformations are also obtained and listed.

In **3** we calculate for the surface S local power series expansions, which express two local non-homogeneous projective coordinates of a point on the surface S as two power series in the other two coordinates and represent the surface S in the neighborhood of an ordinary point on it.

4 is devoted to the derivation of a certain cubic hypercone associated with a point of the surface S.

In **5**, making use of the hyperquadrics having third order contact with the surface S at a point x, we define *a reciprocal correspondence* between certain lines in the tangent plane of the surface S at the point x and certain planes passing through the point x, and call these lines *the canonical lines*

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[1] A projective theory of conjugate nets in ordinary three-dimensional space has been established in a similar way. See E. P. Lane, *A Treatise on Projective Differential Geometry*, The University of Chicago Press, 1942, Chap. VIII.

and these planes *the canonical planes* of the conjugate net N_x at the point x. All canonical lines pass through a point called *the canonical point,* and all canonical planes lie on a quadric hypercone passing through the canonical point called *the canonical quadric hypercone.* The tangent hyperplane of the canonical quadric hypercone at the canonical point is called *the canonical hyperplane.*

6 contains the local power series expansions for the u-, v-curves of the parametric conjugate net N_x on the surface S. These expansions express three local non-homogeneous projective coordinates of a point on each curve as power series in the other coordinate, and represent the curve in the neighborhood of an ordinary point on it. Among the hyperquadrics having third order contact with the surface S at a point x we may determine a pencil of hyperquadrics such that they have fourth order contact with the v-curve and fifth order contact with the u-curve at the point x. The equations of this pencil of hyperquadrics are included at the end of the section.

It is known that as u, v vary the Laplace transformed points x_{-1}, x_1 of an ordinary point x of the surface S with respect to the v-, u-curves of the conjugate net N_x generate two surfaces S_{-1}, S_1, on which the parametric curves also form two conjugate nets N_{-1}, N_1. In the last section geometric characterizations of some special classes of conjugate nets are obtained by studying the pencil of hyperquadrics which have third order contact with the surface $S_{-1}(S_1)$ at the point $x_{-1}(x_1)$ and first order contact with the surface $S_1(S_{-1})$ at the point $x_1(x_{-1})$.

1. Differential equations and integrability conditions. Let us consider in a linear space S_4 of four dimensions a non-developable surface S sustaining a conjugate net N_x with the parametric vector equation,, referred to the net N_x,

$$x = x(u, v). \tag{1.1}$$

The osculating space S_3 of the parametric curve $u(v)$ and the osculating plane of the parametric curve $v(u)$ at an ordinary point x of the surface S intersect in a line $l_1(l_2)$. Let us select on the lines l_1 and l_2 respectively two points y and z, distinct from the point x, and suppose that the coordinates y and z of the points y and z are functions of u, v. Then it can be shown that the coordinates x, y and z of corresponding points x, y and z satisfy a system of linear homogeneous partial differential equations of the form

$$
\begin{aligned}
x_{uu} &= px + \alpha x_u + Lz, \\
x_{uv} &= cx + ax_u + bx_v, \\
x_{vv} &= qx + \delta x_v + Ny, \\
x_{uuu} &= rx + mx_u + \rho y + \beta z, \\
x_{vvv} &= sx + nx_v + \gamma y + \sigma z \qquad (\rho\sigma LN \neq 0),
\end{aligned} \tag{1.2}
$$

in which subscripts indicate partial differentiation and the coefficients are scalar functions of u, v. The second of these equations is merely the Laplace equation for the parametric conjugate net N_x.

It is easy to express the derivatives y_u, y_v as linear combinations of x, x_u, x_v, y, z by using equations (1.2) and

$$(x_{uv})_v = (x_{vv})_u, \quad (x_{vv})_v = x_{vvv};$$

the result is

$$
\begin{aligned}
y_u &= ex + Rx_u + Ex_v + Ay, \\
y_v &= gx + Px_v + By + (\sigma/N)z,
\end{aligned} \tag{1.3}
$$

in which the coefficients are defined by the following equations:

$$
\begin{aligned}
eN &= c_v + ac + bq - c\delta - q_u, & gN &= s - q\delta - q_v, \\
RN &= a_v + a^2 - a\delta - q, & PN &= n - q - \delta_v - \delta^2, \\
EN &= b_v + ab + c - \delta_u, & BN &= \gamma - \delta N - N_v, \\
A &= b - (\log N)_u.
\end{aligned} \tag{1.4}
$$

Analogous expressions for z_u, z_v can be written by making the substitution

$$
\begin{pmatrix} u & x_{-1} & y & a & c & m & p & r & \alpha & \beta & \rho & L & e & g & A & B & E & P & R \\ v & x_1 & z & b & c & n & q & s & \delta & \gamma & \sigma & N & f & h & D & C & F & Q & S \end{pmatrix}. \tag{1.5}
$$

We now proceed to choose for the points y and z two particular covariant points on the lines l_1 and l_2 respectively. To this end we observe that the point X defined by

$$X = y + kx$$

where k is a scalar function of u, v, is on the line l_1. When the point x varies along a curve C_λ of the family represented by the differential equation

$$dv - \lambda du = 0, \tag{1.6}$$

λ being a function of u, v, on the surface S, the point X generates a curve C_X whose tangent at X is determined by X and the point X' given by

$$X' = y_u + y_v\lambda + k(x_u + x_v\lambda) + k'x \qquad (X' = dX/du, \cdots).$$

Expressing X' as a linear combination of x, x_u, x_v, y, z by means of equations (1.3), and equating to zero the coefficients of x_u, x_v therein, we obtain two conditions on the functions k and λ which are necessary and sufficient that the tangent to the curve C_X at the point X lies in the plane $l_1 l_2$, namely,

$$k + R = 0, \qquad E + (P + k)\lambda = 0.$$

Similarly, we can also determine a unique point on the line l_2 and a unique curve of the family (1.6) connected with the point. If we choose these two points respectively for the points y and z, then

$$R = 0, \qquad S = 0, \tag{1.7}$$

and the differential equations of the two curves become

$$Pdv + Edu = 0, \qquad Fdv + Qdu = 0. \tag{1.8}$$

Hereafter it will be supposed that the differential equations (1.2) are in *the canonical form* for which the conditions (1.7) are satisfied.

The integrability conditions of equations (1.2) are found by the usual method from the equations

$$(y_u)_v = (y_v)_u, \qquad (z_u)_v = (z_v)_u$$

and the fact that the points x, x_u, x_v, y, z are linearly independent. These conditions are

$$\begin{aligned} &e_v + gA + qE = g_u + eB + cP + h\sigma/N, \quad g + aP + \sigma Q/N = 0, \\ &e + E_v + \delta E + AP = P_u + bP + BE, \qquad A_v + EN = B_u + \rho\sigma/L \\ &\sigma_u = \sigma(b - c), \end{aligned} \tag{1.9}$$

and the analogous ones obtainable therefrom by the substitution (1.5).

Making use of equations (1.4), the fourth of equations (1.9) and the substitution (1.5) we obtain the equation

$$(2b + \beta/L)_v = (2a + \gamma/N)_u. \tag{1.10}$$

It follows that there exists a function θ of u, v which is defined, except for an arbitrary additive constant, as a solution of the differential equations

$$\theta_u = 2b + \beta/L - (\log LN)_u, \qquad \theta_v = 2a + \gamma/N - (\log LN)_v. \tag{1.11}$$

Accordingly, the following formula is valid:

$$(x, x_u, x_v, y, z) = e^{\theta}, \tag{1.12}$$

where a determinant is indicated by writing only a typical row within parentheses.

2. Transformations and invariants. Let us consider the group of transformations on the coordinates x, y, z and the parameters u, v:

$$x = \lambda\bar{x}, \quad y = \mu\bar{y}, \quad z = \nu\bar{z} \qquad (\lambda\mu\nu \neq 0), \tag{2.1}$$

$$\bar{u} = U(u), \quad \bar{v} = V(v) \qquad (U'V' \neq 0), \tag{2.2}$$

where λ, μ, ν are scalar functions of u, v and the accent denotes differentiation with respect to the appropriate variable.

The effect of the transformation (2.1) on the system of equations (1.2) is to produce another system of equations of the same form whose coefficients, indicated by dashes, are given by the following formulas and the analogous ones:

$$\begin{aligned}
&\bar{p} = (1/\lambda)(p\lambda + \alpha\lambda_u - \lambda_{uu}), \quad \bar{\alpha} = \alpha - 2(\lambda_u/\lambda), \quad \bar{L} = (\nu/\lambda)L,\\
&\bar{c} = (1/\lambda)(c\lambda + a\lambda_u + b\lambda_v - \lambda_{uv}), \quad \bar{a} = a - \lambda_v/\lambda,\\
&\bar{r} = r + (1/\lambda)(m - 3p)\lambda_u + 3(\lambda_u/\lambda^2)(\lambda_{uu} - \alpha\lambda_u) - \lambda_{uuu}/\lambda,\\
&\bar{m} = m - (3/\lambda)(\alpha\lambda_u + \lambda_{uu}) + 6(\lambda_u^2/\lambda^2), \quad \bar{\rho} = (\mu/\lambda)\rho,\\
&\bar{\beta} = (\nu/\lambda^2)(\beta\lambda - 3L\lambda_u).
\end{aligned} \tag{2.3}$$

The effect of the transformation (2.2) on the system of equations (1.2) is to produce another system of equations of the same form whose coefficients, indicated by stars, are given by the following formulas and the analogous ones:

$$\begin{aligned}
&p^* = (1/U'^2)p, \quad \alpha^* = (1/U')(\alpha - U''/U'), \quad L^* = (1/U'^2)L,\\
&c^* = (1/U'V')c, \quad a^* = (1/V')a,\\
&r^* = (1/U'^3)[r - 3(U''/U')p],\\
&m^* = (1/U'^2)[m - (1/U')(3\alpha U'' + U''') + 3(U''/U')^2],\\
&\rho^* = (1/U'^3)\rho, \quad \beta^* = (1/U'^3)[\beta - 3(U''/U')L].
\end{aligned} \tag{2.4}$$

From equations (2.3), (2.4) and the substitution (1.5) we may obtain, after some calculation, the following functions which are absolute invariants, under the transformation (2.1), and relative invariants, under the transformation (2.2), of the system of equations (1.2):

$$\begin{aligned}
&H = c + ab - a_u, \qquad \mathfrak{H} = 3(b - \alpha) + \beta/L,\\
&I = L/\sigma, \qquad \mathfrak{M} = 4b - 2\alpha + \phi_u, \qquad \mathfrak{S} = EN,\\
&\mathfrak{T} = r + bm - b^3 - 3bb_u - b_{uu}, \qquad \mathfrak{A} = 3(2b - \alpha) + [\log(\rho/N)]_u,\\
&\mathfrak{B} = -3m + 3(6b^2 - 10b\alpha + 5\alpha^2) + 6b_u + (\beta/L)(8b - 7\alpha)\\
&\qquad + \beta^2/L^2 + (\beta/L)_u,
\end{aligned} \tag{2.5}$$

where $\phi = \log I$.

The effect of the transformation (2.2) on each of these invariants is given, with self-explanatory notation, by the following formulas:

$$
\begin{aligned}
&H^* = (1/U'V')H, && \mathfrak{H}^* = (1/U')\mathfrak{H}, \\
(2.6)\quad &I^* = (V'^3/U'^2)I, && \mathfrak{M}^* = (1/U')\mathfrak{M}, && \mathfrak{S}^* = (1/U'V')\mathfrak{S}, \\
&\mathfrak{F}^* = (1/U'^3)\mathfrak{F}, && \mathfrak{A}^* = (1/U')\mathfrak{A}, && \mathfrak{B}^* = (1/U'^2)\mathfrak{B}.
\end{aligned}
$$

Analogous invariants can be written by using the substitution (1.5) and

$$
(2.7)\qquad \begin{pmatrix} H & I & \mathfrak{A} & \mathfrak{B} & \mathfrak{F} & \mathfrak{H} & \phi & \mathfrak{M} & \mathfrak{S} \\ K & J & \mathfrak{D} & \mathfrak{C} & \mathfrak{G} & \mathfrak{K} & \psi & \mathfrak{N} & \mathfrak{T} \end{pmatrix}.
$$

Two of the most important covariants of the system of equations (1.2) under the total transformation (2.1), (2.2) are the functions x_1, x_{-1} defined by the formulas

$$
(2.8)\qquad x_1 = x_v - ax, \qquad x_{-1} = x_u - bx.
$$

The points x_1, x_{-1} are called, respectively, *the first and minus-first Laplace transformed points* of the point x with respect to the parametric conjugate net N_x on the surface S, or the *ray-points* of the u-, v-curves at the point x.

3. Power series expansions for a surface. Let us consider an ordinary point x with curvilinear coordinates u, v on an integral surface S of equations (1.2) with the conditions (1.7). The coordinates X, where

$$
X = x(u + \Delta u, v + \Delta v),
$$

of any point X near the point x on the surface S are given by the Taylor's series

$$
\begin{aligned}
(3.1)\quad X = x &+ x_u\Delta u + x_v\Delta v \\
&+ \tfrac{1}{2}(x_{uu}\Delta u^2 + 2x_{uv}\Delta u\Delta v + x_{vv}\Delta v^2) + \cdots,
\end{aligned}
$$

in which the increments Δu and Δv correspond to displacement on the surface S from the point x to the point X. By means of eequations (1.2), (1.3), the equations obtained therefrom by differentiation and the substitution (1.5), it is possible to express every derivative of x uniquely as a linear combination of x, x_u, x_v, y, z.

If the points x, x_{-1}, x_1, y, z are used as the vertices of the pyramid of reference, with unit point suitably chosen, then any point given by an expression of the form

$$
(3.2)\qquad \xi_1 x + \xi_2 x_{-1} + \xi_3 x_1 + \xi_4 y + \xi_5 z
$$

has local coordinates proportional to $\xi_1, \cdots, \xi_5$. Replacing the derivatives x_u, x_v in the foregoing formulas by their equivalents as given by equations (2.8), we are able easily to express derivatives of x of any required order as linear combinations of x, x_{-1}, x_1, y, z. Then substituting these expressions for the derivatives of x in the Taylor's series (3.1) and making use of equations (1.4), (2.5), the substitution (2.7) and the definitions

$$\xi = \xi_2/\xi_1, \qquad \eta = \xi_3/\xi_1, \qquad \zeta = \xi_4/\xi_1, \qquad \tau = \xi_5/\xi_1, \tag{3.3}$$

we may obtain *the power series expansions for the non-homogeneous local coordinates* ζ, τ *of the point* X *in terms of the other two coordinates* ξ, η:

$$\begin{aligned} \zeta &= \tfrac{1}{2}N\eta^2 + \tfrac{1}{6}\rho\xi^3 + \tfrac{1}{6}N\mathfrak{R}\eta^3 + (\rho/24)(\mathfrak{A} + \mathfrak{H})\xi^4 \\ &\quad + \tfrac{1}{6}NK\xi\eta^3 + (1/24)N\mathfrak{C}\eta^4 + \cdots, \\ \tau &= \tfrac{1}{2}L\xi^2 + \tfrac{1}{6}L\mathfrak{H}\xi^3 + \tfrac{1}{6}\sigma\eta^3 + (1/24)L\mathfrak{B}\xi^4 \\ &\quad + \tfrac{1}{6}LH\xi^3\eta + (\sigma/24)(\mathfrak{D} + \mathfrak{K})\eta^4 + \cdots. \end{aligned} \tag{3.4}$$

The first of equations (3.4) and $\tau = 0$ demonstrate that *the projection of the surface* S *from the point* z *into the space* S_3 $xx_{-1}x_1y$ *has coincident asymptotic tangents at the point* x. A similar statement holds for the second of equations (3.4) and $\zeta = 0$.

4. A certain cubic hypercone. In this section we shall derive a certain cubic hypercone by considering a general curve C_λ represented by the differential equation (1.6). The first three derivatives of x with respect to the independent variable u along the curve C_λ are found to be

$$\begin{aligned} x' &= x_u + x_v\lambda \qquad (x' = dx/du, \cdots), \\ x'' &= G + x_v\lambda', \\ x''' &= H + 3(x_{uv} + x_{vv}\lambda)\lambda' + x_v\lambda'', \end{aligned} \tag{4.1}$$

where G, H are defined by placing

$$\begin{aligned} G &= x_{uu} + 2x_{uv}\lambda + x_{vv}\lambda^2, \\ H &= x_{uuu} + 3x_{uuv}\lambda + 3x_{uvv}\lambda^2 + x_{vvv}\lambda^3. \end{aligned} \tag{4.2}$$

The space $S(2,1)$ of the surface S in the direction of the curve C_λ at the point x, which is defined as the space of least dimensionality containing the osculating planes at the point x of all curves lying on the surface S and tangent to the curve C_λ at the point x, is a hyperplane determined by the points

$$x, x_u, x_v, G. \tag{4.3}$$

The differential equation of the two-parameter family of the quasi-asymptotic curves of Bompiani [2] at the point x of the surface S now takes the form

$$3LN\lambda\lambda' + \rho L + N(3bL - \beta)\lambda^2 + L(\gamma - 3aN)\lambda^3 - \sigma N\lambda^5 = 0. \tag{4.4}$$

Since the equation of the space $S(2,1)$ in local coordinates is

$$L\xi_4 - N\lambda^2\xi_5 = 0, \tag{4.5}$$

we may easily verify that *every hyperplane through the tangent plane at the point x of the surface S is the osculating space S_3 of two of these quasi-asymptotic curves at the point x, whose directions separate the u-, v-tangents at the point x harmonically.*[3]

The tangent line of the curve C_λ and the plane of intersection of the two osculating spaces S_3 of the u-, v-curves at the point x determine a hyperplane with the equation

$$\xi_3 - \lambda\xi_2 = 0. \tag{4.6}$$

As the direction (1.6) *varies in the tangent plane of the surface S at the point x, the locus of the plane of intersection of the two hyperplanes* (4.5), (4.6) *is a cubic hypercone with vertex at the point x,* whose equation is found to be

$$L\xi_2{}^2\xi_4 - N\xi_3{}^2\xi_5 = 0. \tag{4.7}$$

This cubic hypercone has $xx_{-1}z$, xx_1y for its double planes and passes through the tangent plane $xx_{-1}x_1$.

5. Canonical configurations for a surface. There is a four-parameter family of hyperquadrics having third order contact with the surface S at a point x. The equation of a general one of these hyperquadrics is obtained by writing the equation of the most general non-singular hyperquadric and demanding that the series (3.4) satisfy this equation identically in ξ, η as far as the terms of the third degree. The result can be written in the form

$$\begin{aligned} &\tfrac{1}{2}La_{45}\xi^2 + (1/3L)(\rho a_{35} + L\mathfrak{H}a_{45})\xi\tau + \tfrac{1}{2}Na_{35}\eta^2 \\ &\quad + (1/3N)(N\mathfrak{R}a_{35} + \sigma a_{45})\eta\zeta + a_{33}\zeta^2 + a_{34}\zeta\tau \\ &\quad + a_{44}\tau^2 - a_{35}\xi - a_{45}\tau = 0, \end{aligned} \tag{5.1}$$

---

[2] E. Bompiani, "Sopra alcune estensioni dei teoremi di Meusnier di Eulero," *Atti della Reale Accademia delle Scienze di Torino*, vol. 48 (1912), p. 404; or see E. P. Lane, *loc. cit.*[1], p. 414.

[3] E. Bompiani, *loc. cit.*[2]

where the coefficients a_{ij} are parameters. The polar hyperplanes of the points x_{-1}, x_1 with respect to any hyperquadric of the family (5.1) have respectively the equations

$$La_{45}\xi + (1/3L)(\rho a_{35} + L\mathfrak{H}a_{45})\tau = 0, \tag{5.2}$$

$$Na_{35}\eta + (1/3N)(N\mathfrak{K}a_{35} + \sigma a_{45})\zeta = 0. \tag{5.3}$$

Let us now consider in the tangent plane $xx_{-1}x_1$ two directions which pass through the point x and separate the u-, v-tangents harmonically. The equations of these directions may be written in the form

$$\eta^2 - \lambda^2\xi^2 = 0, \qquad \zeta = 0, \qquad \tau = 0, \tag{5.4}$$

where λ is arbitrary. If the hyperquadric (5.1) cuts the tangent plane $xx_{-1}x_1$ in the directions (5.4), the polar hyperplanes (5.2), (5.3) become respectively

$$N\lambda^2\xi + (1/3L)(N\mathfrak{H}\lambda^2 - \rho)\tau = 0. \tag{5.5}$$

$$L\eta + (1/3N)(L\mathfrak{K} - \sigma\lambda^2)\zeta = 0. \tag{5.6}$$

As the tangents (5.4) *vary about the point* x *the plane of intersection of the two hyperplanes* (5.5), (5.6) *generates a quadric hypercone with vertex at the point* x, whose equation is

$$[\xi + (\mathfrak{H}/3L)\tau][\eta + (\mathfrak{K}/3N)\zeta] = (\rho\sigma/9L^2N^2)\zeta\tau. \tag{5.7}$$

It is evident that *the polar plane of the tangent plane* $xx_{-1}x_1$ *with respect to the hypercone* (5.7) *is*

$$\xi + (\mathfrak{H}/3L)\tau = 0, \qquad \eta + (\mathfrak{K}/3N)\zeta = 0. \tag{5.8}$$

If the line yz lies on the hyperquadrics (5.1), then $a_{33} = a_{34} = a_{44} = 0$ and therefore we obtain a pencil of hyperquadrics:

$$\begin{aligned} a_{35}[\zeta - (\rho/3L)\xi\tau - (N/2)\eta^2 - (\mathfrak{K}/3)\eta\zeta] \\ + a_{45}[\tau - (\mathfrak{H}/3)\xi\tau - (L/2)\xi^2 - (\sigma/3N)\eta\zeta] = 0. \end{aligned} \tag{5.9}$$

In this pencil there is a unique pair, one passing through the points x_{-1}, z and the other through the points x_1, y, whose equations are respectively

$$\zeta - (\rho/3L)\xi\tau - (N/2)\eta^2 - (\mathfrak{K}/3)\eta\zeta = 0, \tag{5.10}$$

$$\tau - (\mathfrak{H}/3)\xi\tau - (L/2)\xi^2 - (\sigma/3N)\eta\zeta = 0. \tag{5.11}$$

The polar hyperplanes p_2, p_1 of the points x_1, x_{-1} with respect to the hyperquadrics (5.10), (5.11), respectively, are given by equations (5.8). More-

over, the polar hyperplanes of the points y, z with respect to the hyperquadrics (5.10), (5.11), are, respectively,

$$3 - \mathfrak{R}\eta = 0, \qquad 3 - \mathfrak{H}\xi = 0, \tag{5.12}$$

which intersect the v-, u-tangents respectively in the points

$$\mathfrak{R}x + 3x_1, \qquad \mathfrak{H}x + 3x_{-1}. \tag{5.13}$$

Let P_2, P_1 be the harmonic conjugate points of the two points (5.13) with respect to the points x, x_1 and x, x_{-1} respectively, then on the lines xx_{-1}, xx_1 we may determine two points Q_1, Q_2 such that the cross ratios

$$(xx_{-1}, P_1Q_1) = 3k, \qquad (xx_1, P_2Q_2) = 3k, \tag{5.14}$$

where k is an arbitrary constant independent of u, v. The equations of the line Q_1Q_2 are of the form

$$1 + k\mathfrak{H}\xi + k\mathfrak{R}\eta = 0, \qquad \zeta = 0, \qquad \tau = 0. \tag{5.15}$$

In like manner, we can determine two hyperplanes q_1, q_2 such that the cross ratios

$$(xx_{-1}x_1y, xx_1yz, p_1, q_1) = 3k, \qquad (xx_{-1}x_1z, xx_{-1}yz, p_2, q_2) = 3k. \tag{5.16}$$

The plane of intersection of the two hyperplanes q_1, q_2 is given by the equations

$$L\xi + k\mathfrak{H}\tau = 0, \qquad N\eta + k\mathfrak{R}\zeta = 0. \tag{5.17}$$

The line Q_1Q_2 and the plane q_1q_2 will be said to be *reciprocal* with respect to the net N_x. The line Q_1Q_2 will be called *a canonical line* and the plane q_1q_2 *a canonical plane* of the net N_x at the point x. All canonical lines pass through the point

$$(0, \mathfrak{R}, -\mathfrak{H}, 0, 0), \tag{5.18}$$

which we shall call *the canonical point* at the point x of the net N_x. All canonical planes lie on the quadric hypercone

$$L\mathfrak{R}\xi\zeta - N\mathfrak{H}\eta\tau = 0, \tag{5.19}$$

which we shall call *the canonical quadric hypercone* at the point x of the net N_x. The tangent hyperplane of the canonical quadric hypercone at the canonical point will be called *the canonical hyperplane* at the point x of the net N_x; its equation is

$$L\mathfrak{R}^2\zeta + N\mathfrak{H}^2\tau = 0. \tag{5.20}$$

If the plane of intersection of the two polar hyperplanes (5.2), (5.3)

is a general canonical plane (5.17), *then the conjugate net* N_x *is restricted by the condition*

$$(5.21)\qquad (3k-1)^2 IJ\mathfrak{H}\mathfrak{K} = 1.$$

Moreover, we may determine two pairs of tangents through the point x such that the plane of intersection of the two polar hyperplanes (5.5), (5.6) is a general canonical plane (5.17). The equations of these tangents are easily found to be

$$(5.22)\qquad (1-3k)N\mathfrak{H}\eta^2 - \rho\xi^2 = 0, \qquad \zeta = 0, \qquad \tau = 0,$$

$$(5.23)\qquad \sigma\eta^2 - (1-3k)L\mathfrak{K}\xi^2 = 0, \qquad \zeta = 0, \qquad \tau = 0.$$

6. Curves of a conjugate net. In this section we shall study the u-, v-curves of the parametric conjugate net N_x on an integral surface S of equations (1.2). For this purpose we first consider a point X near a point x and on the u-curve through x, whose coordinates are given by the Taylor's series in the increment Δu corresponding to displacement along the u-curve from the point x to the point X. It is easy to express *the non-homogeneous local coordinates* η, ζ, τ *of the point* X *as power series in the other coordinate* ξ, *namely,*

$$\begin{aligned}
\eta &= (\mathfrak{S}/24J)\xi^4 + \cdots,\\
\zeta &= \tfrac{1}{6}\rho\xi^3 + (1/24)\rho(\mathfrak{A}+\mathfrak{H})\xi^4\\
&\quad + (1/120)\rho[\mathfrak{A}_u - \mathfrak{H}_u + \mathfrak{A}^2 + \mathfrak{A}\mathfrak{H} - 2\mathfrak{H}^2 + 3\mathfrak{B}\\
&\quad + \tfrac{1}{2}(\mathfrak{A}-\mathfrak{H})(\mathfrak{M}-\phi_u)]\xi^5 + \cdots,\\
\tau &= \tfrac{1}{2}L\xi^2 + \tfrac{1}{6}L\mathfrak{H}\xi^3 + (1/24)L\mathfrak{B}\xi^4\\
&\quad + (1/120)L\{\mathfrak{B}_u - (10/3)\mathfrak{H}(\mathfrak{H}_u + \mathfrak{H}^2) + (13/3)\mathfrak{B}\mathfrak{H} + 6\mathfrak{F}\\
&\quad + [\mathfrak{B} - (5/3)\mathfrak{H}^2](\mathfrak{M}-\phi_u)\}\xi^5 + \cdots.
\end{aligned}\qquad (6.1)$$

In like manner, *the power series expansions of the* v*-curve in the neighborhood of the point* x *are*

$$\begin{aligned}
\xi &= (\mathfrak{T}/24I)\eta^4 + \cdots,\\
\zeta &= \tfrac{1}{2}N\eta^2 + \tfrac{1}{6}N\mathfrak{K}\eta^3 + (1/24)N\mathfrak{C}\eta^4\\
&\quad + (1/120)N\{\mathfrak{C}_v - (10/3)\mathfrak{K}(\mathfrak{K}_v + \mathfrak{K}^2) + (13/3)\mathfrak{C}\mathfrak{K} + 6\mathfrak{G}\\
&\quad + [\mathfrak{C} - (5/3)\mathfrak{K}^2](\mathfrak{N}-\psi_v)\}\eta^5 + \cdots,\\
\tau &= \tfrac{1}{6}\sigma\eta^3 + (1/24)\sigma(\mathfrak{D}+\mathfrak{K})\eta^4\\
&\quad + (1/120)\sigma[\mathfrak{D}_v - \mathfrak{K}_v + \mathfrak{D}^2 + \mathfrak{D}\mathfrak{K} - 2\mathfrak{K}^2 + 3\mathfrak{C}\\
&\quad + \tfrac{1}{2}(\mathfrak{D}-\mathfrak{K})(\mathfrak{N}-\psi_v)]\eta^5 + \cdots.
\end{aligned}\qquad (6.2)$$

From the expansions (6.1), (6.2) it follows that the conditions that a general hyperquadric (5.1) has fourth order contact with both the u-, v-curves at the point x are

$$(6.3)\qquad \begin{aligned} a_{33} &= (1/18N^2)[N(3\mathfrak{C}-4\mathfrak{K}^2)a_{35}+\sigma(3\mathfrak{D}-\mathfrak{K})a_{45}],\\ a_{44} &= (1/18L^2)[\rho(3\mathfrak{A}-\mathfrak{H})a_{35}+L(3\mathfrak{B}-4\mathfrak{H}^2)a_{45}]. \end{aligned}$$

This hyperquadric has fifth order contact with the u-curve at the point x in case

$$(6.4)\qquad a_{34} = (1/10L)A_1a_{35}+(1/10\rho)A_2a_{45},$$

where we have placed

$$(6.5)\qquad \begin{aligned} A_1 &= \mathfrak{A}_u-\mathfrak{H}_u+\mathfrak{A}^2-(7/3)\mathfrak{A}\mathfrak{H}-(8/9)\mathfrak{H}^2+(4/3)\mathfrak{B}\\ &\quad +\tfrac{1}{2}(\mathfrak{A}-\mathfrak{H})(\mathfrak{M}-\phi_u),\\ A_2 &= \mathfrak{B}_u-(10/3)\mathfrak{H}\mathfrak{H}_u+(10/9)\mathfrak{H}^3-(2/3)\mathfrak{B}\mathfrak{H}+6\mathfrak{F}\\ &\quad +[\mathfrak{B}-(5/3)\mathfrak{H}^2](\mathfrak{M}-\phi_u). \end{aligned}$$

Thus in the family of hyperquadrics having third order contact with the surface S at a point x, we obtain a pencil having fourth order contact with the v-curve and fifth order contact with the u-curve at the point x. In particular, a unique hyperquadric in this pencil may be determined to pass through the point y or z or to have fifth order contact with the v-curve at the point x.

7. Laplace transformed surfaces S_{-1}, S_1. It is known that as u, v vary the Laplace transformed points x_{-1}, x_1 of an ordinary point x of the surface S with respect to the v-, u-curves of the conjugate net N_x generate two surfaces S_{-1}, S_1, on which the parametric curves also form two conjugate nets N_{-1}, N_1. As usual, we call the surfaces S_{-1}, S_1 and the nets N_{-1}, N_1, respectively, the minus-first and first Laplace transformed surfaces and nets of N_x. In this section we shall first find the power series expansions of the surfaces S_{-1}, S_1 at the points x_{-1}, x_1.

From system (1.2), equations (1.3), (2.8) and the substitution (1.5) by differentiation and substitution, any derivative of x_{-1} can be expressed as a linear combination of x, x_u, x_v, y, z. In particular, one obtains

$$(7.1)\qquad \begin{aligned} x_{-1uu} &= (r-b_{uu}-bp)x+(m-2b_u-b\alpha)x_u+\rho y+(\beta-bL)z,\\ x_{-1uv} &= (c_u-b_{uv}+ap)x+(a_u-b_v+c+a\alpha)x_u+aLz,\\ x_{-1vv} &= (c_v-b_{vv}+ac)x+(a_v+a^2)x_u+Kx_v,\\ x_{-1uuu} &= (*)x+(*)x_u+\rho Ex_v+\rho(\beta/L+\rho_u/\rho-N_u/N)y+(*)z,\\ x_{-1uuv} &= (*)x+(*)x_u+a\rho y+(*)z,\\ x_{-1uvv} &= (*)x+(*)x_u+(K_u+bK)x_v+(*)z,\\ x_{-1vvv} &= (*)x+(*)x_u+[2K_v+K(a+\delta)]x_v+NKy, \end{aligned}$$

where $(*)$ denotes terms immaterial for our purpose.

The coordinates of any point X near the point x_{-1} on the surface S_{-1} can be represented by the Taylor's expansion as power series in the increments Δu, Δv corresponding to displacement on S_{-1} from x_{-1} to the point X. From equations (7.1) we find that the local coordinates $\xi_1, \cdots, \xi_5$ of the point X are given by the expansions

$$
\begin{aligned}
\xi_1 &= K\Delta v + \tfrac{1}{2}\mathfrak{F}\Delta u^2 + (K_u + bK)\Delta u\Delta v + \tfrac{1}{2}(K_v + 2aK)\Delta v^2 + \cdots, \\
\xi_2 &= 1 + (\alpha - b)\Delta u + a\Delta v + \tfrac{1}{2}(m - 2b_u - b\alpha)\Delta u^2 \\
&\quad + (a_u - b_u + c + a\alpha)\Delta u\Delta v + \tfrac{1}{2}(a_v + a^2)\Delta v^2 + \cdots, \\
\xi_3 &= \tfrac{1}{2}K\Delta v^2 + \tfrac{1}{6}\rho E\Delta u^3 + \tfrac{1}{2}(K_u + bK)\Delta u\Delta v^2 \\
&\quad + \tfrac{1}{6}[2K_v + K(a + \delta)]\Delta v^3 + \cdots, \\
\xi_4 &= \tfrac{1}{2}\rho\Delta u^2 + \tfrac{1}{6}\rho(\beta/L + \rho_u/\rho - N_u/N)\Delta u^3 + \tfrac{1}{2}a\rho\Delta u^2\Delta v \\
&\quad + \tfrac{1}{6}NK\Delta v^3 + \cdots, \\
\xi_5 &= L\Delta u + \tfrac{1}{2}(\beta - bL)\Delta u^2 + aL\Delta u\Delta v + \cdots.
\end{aligned}
\tag{7.2}
$$

By means of the series (7.2) it follows that a hyperquadric with the general equation

$$
\sum_{i,k=1}^{5} a_{ik}\xi_i\xi_k = 0 \qquad (a_{ik} = a_{ki}), \tag{7.3}
$$

has second order contact with the surface S_{-1} at the point x_{-1} in case

$$
a_{22} = a_{25} = a_{12} = a_{15} = 0, \qquad a_{55} = -(\rho/L^2)a_{24}, \qquad a_{23} = -Ka_{11}. \tag{7.4}
$$

Similarly, the hyperquadric (7.3) has second order contact with the surface S_1 at the point x_1 if, and only if,

$$
a_{33} = a_{13} = a_{34} = a_{14} = 0, \qquad a_{44} = -(\sigma/N^2)a_{35}, \qquad a_{23} = -Ha_{11}. \tag{7.5}
$$

Thus we reach the following theorem:[4]

Conjugate nets with equal and non-zero Laplace-Darboux invariants H, K in space S_4 are characterized by the property that there exists a proper hyperquadric (and therefore ∞^3 such hyperquadrics) having second order contact with both the Laplace transformed surfaces S_{-1}, S_1 at the Laplace transformed points x_{-1}, x_1 respectively.

Making use of the conditions (7.4) and the first three of the conditions (7.5) and imposing further on equation (7.3) the conditions that it be satisfied by the series (7.2) identically in terms of the third degree of Δu, Δv, it is easy to obtain *the equations of a pencil of hyperquadrics having*

[4] This theorem is not true for conjugate nets in ordinary space. See the author's paper, "New geometrical characterizations of some special conjugate nets," *Duke Mathematical Journal*, vol. 12 (1945), p. 252.

third order contact with the surface S_{-1} at the point x_{-1} and first order contact with the surface S_1 at the point x_1, namely,

$$(7.6)\quad \begin{aligned}&\xi_1^2 - 2(\mathfrak{F}/\rho)\xi_1\xi_4 - 2K\xi_2\xi_3 - (1/N)[2K_v + K(\mathfrak{N} - \psi_v)]\xi_2\xi_4\\ &- (1/L)[2K_u + K(\mathfrak{M} - \phi_u)]\xi_3\xi_5\\ &- (1/3LN)\{(2\mathfrak{H} - \mathfrak{A})[2K_v + K(\mathfrak{N} - \psi_v)] - 2K\mathfrak{S}\}\xi_4\xi_5\\ &+ (\rho/2L^2N)[2K_v + K(\mathfrak{N} - \psi_v)]\xi_5^2 + k_4\xi_4^2 = 0,\end{aligned}$$

where k_4 is a parameter. There is also *a pencil of hyperquadrics having third order contact with the surface S_1 at the point x_1 and first order contact with the surface S_{-1} at the point x_{-1}.* The equation of this pencil can be obtained in a way similar to the foregoing, or else can be written immediately by making the substitutions (1.5), (2.7) and the necessary symmetrical interchanges of the subscripts; the result is

$$(7.7)\quad \begin{aligned}&\xi_1^2 - 2(\mathfrak{G}/\sigma)\xi_1\xi_5 - 2H\xi_2\xi_3 - (1/N)[2H_v + H(\mathfrak{N} - \psi_v)]\xi_2\xi_4\\ &- (1/L)[2H_u + H(\mathfrak{M} - \phi_u)]\xi_3\xi_5\\ &- (1/3LN)\{(2\mathfrak{R} - \mathfrak{D})[2H_u + H(\mathfrak{M} - \phi_u) - 2H\mathfrak{T}\}\xi_4\xi_5\\ &+ (\sigma/2LN^2)[2H_u + H(\mathfrak{M} - \phi_u)]\xi_4^2 + k_5\xi_5^2 = 0,\end{aligned}$$

where k_5 is a parameter. If a unique hyperquadric in the pencil (7.6) or (7.7) is desired, we may choose the one that passes through the point y or z. For these hyperquadrics we have $k_4 = 0$, $k_5 = 0$ respectively.

New geometric characterizations of the conjugate net N_x in some special cases may be deduced by studying the hyperquadrics (7.6), (7.7). In the first place, it is clear that the tangent plane at the point x of the surface S intersects the pencils (7.6), (7.7) in the conics of Koenigs, which coincide in case the conjugate net N_x has equal Laplace-Darboux invariants H, K. Two hyperquadrics (7.6), (7.7) with general k_4, k_5 determine a pencil, in which there is one passing through the point x. The tangent hyperplanes of this hyperquadric at the points x, x_{-1}, x_1 are respectively

$$(7.8)\quad \sigma\mathfrak{F}\xi_4 - \rho\mathfrak{G}\xi_5 = 0,$$

$$(7.9)\quad (H - K)\xi_3 + (1/N)[H_v - K_v + \tfrac{1}{2}(H - K)(\mathfrak{N} - \psi_v)]\xi_4 = 0,$$

$$(7.10)\quad (H - K)\xi_2 + (1/L)[H_u - K_u + \tfrac{1}{2}(H - K)(\mathfrak{M} - \phi_u)]\xi_5 = 0.$$

The hyperplane (7.9) or (7.10) is indeterminate if, and only if, $H = K$. Thus we conclude that *a necessary and sufficient condition that the hyperquadric passing through the point x in the pencil determined by any two hyperquadrics* (7.6), (7.7) *be a quadric hypercone with a vertex at the point x_{-1} or x_1 (and therefore with a line of vertices at $x_{-1}x_1$) is $H = K$.*

The tangent hyperplanes of any hyperquadrics (7.6), (7.7) at the point x_{-1} are respectively

$$2K\xi_3 + (1/N)[2K_v + K(\mathfrak{N} - \psi_v)]\xi_4 = 0, \tag{7.11}$$

$$2H\xi_3 + (1/N)[2H_v + H(\mathfrak{N} - \psi_v)]\xi_4 = 0, \tag{7.12}$$

which coincide in case

$$H/K = U, \tag{7.13}$$

where U is a function of u alone. Moreover, the hyperplanes (7.11), (7.12) are separated harmonically by the hyperplanes $xx_{-1}yz$, $xx_{-1}x_1z$ if, and only if,

$$(\log HK)_v + \mathfrak{N} - \psi_v = 0. \tag{7.14}$$

Similarly, the tangent hyperplanes of any hyperquadrics (7.6), (7.7) at the point x_1 are respectively

$$2K\xi_2 + (1/L)[2K_u + K(\mathfrak{M} - \phi_u)]\xi_5 = 0, \tag{7.15}$$

$$2H\xi_2 + (1/L)[2H_u + H(\mathfrak{M} - \phi_u)]\xi_5 = 0, \tag{7.16}$$

which coincide in case

$$H/K = V, \tag{7.17}$$

where V is a function of v alone. Furthermore, the hyperplanes (7.15), (7.16) are separated harmonically by the hyperplanes xx_1yz, $xx_{-1}x_1y$ if, and only if,

$$(\log HK)_u + \mathfrak{M} - \phi_u = 0. \tag{7.18}$$

If the conditions (7.13), (7.17) hold simultaneously, then

$$H/K = \text{const.} \tag{7.19}$$

Moreover, if the conditions (7.14), (7.18) hold simultaneously, then the conjugate net N_x belongs to the class restricted by the condition

$$(\mathfrak{N} - \psi_v)_u = (\mathfrak{M} - \phi_u)_v. \tag{7.20}$$

Finally, we shall observe that in the pencil determined by any two hyperquadrics (7.6), (7.7) there is one whose tangent hyperplane at the point $x_{-1}(x_1)$ passes through the point $x_1(x_{-1})$. The polar hyperplane of the point x with respect to this hyperquadric is

$$(H - K)\xi_1 - (H\mathfrak{F}/\rho)\xi_4 + (K\mathfrak{G}/\sigma)\xi_5 = 0. \tag{7.21}$$

MICHIGAN STATE COLLEGE AND
NATIONAL UNIVERSITY OF CHEKIANG.

POLYNOMIAL MATRICES IN SEVERAL VARIABLES.*

By Ernst Snapper.

Introduction and Summary. Let $P[x_1, \cdots, x_n]$ denote the ring of polynomials in n variables $x_1, \cdots, x_n$ whose coefficients belong to the commutative field P. The purpose of this paper is to develop the theory of an $s \times m$ matrix A whose elements belong to $P[x_1, \cdots, x_n]$. The theory is a generalization of the case $n = 1$ which was treated in Part I of [1] (square brackets refer to the references) and is based on the module theory of Part II of [1]. In Part I of the present paper it is shown how a system S of partial, homogeneous, linear differential equations with constant coefficients in n independent variables and m dependent variables arises from A if the variables $x_1, \cdots, x_n$ are replaced by the differential operators $\partial/\partial t_1, \cdots, \partial/\partial t_n$. The exponential solutions of S are then investigated by the use of the module theory of the row space of A. In Part II of this paper it is shown how the theory of the Hilbert characteristic function can be extended to a polynomial module M such as the row space or the column space of A. This extension gives rise to the notion of the *degrees* of M. The relationship between these degrees and the associated primes of M and the length of the primary components of M, as defined in [1], is derived. Furthermore, the norm and elementary divisor of M, as defined in [2], are studied. It is proved that the degrees of the primary factors of the norm of M are equal to the degrees of M and that, if the characteristic of P is zero, the multiplicities of the roots of the elementary divisor of M are equal to the $\mathfrak{p}$-exponents of a Noether decomposition of M. It is shown that the exponents which occur in the theory of the exponential solutions of S, as developed in Part I of this paper, are equal to the degrees and $\mathfrak{p}$-exponents of the row space of A and hence that, as in the case $n = 1$, the norm and the elementary divisor of the row space of A determine the algebraic properties of the exponential solutions of S. Finally, a system of algebraic, linear equations

$$\sum_{j=1}^{m} a_{ij} Z_j = \gamma_i, \qquad i = 1, \cdots, s, \tag{1}$$

* Received October 4, 1945.

where α_{ij} and γ_i are given elements of $P[x_1, \cdots, x_n]$, is studied. A criterion for the solvability of (1) by elements $Z_j \in P[x_1, \cdots, x_n]$ is derived in terms of the degrees of the column space of A. The equivalence of this criterion with theorems 2.72 of [1] and 5.3 of [2] has been established.

I. Systems of Partial Differential Equations.

1.1. The vector space over the algebra of exponential and partial differentiation. This section is a generalization of Section 1 of [3] to systems of partial differential equations. Let $P[t_1, \cdots, t_n]$ be a polynomial ring where P is a field of characteristic zero and where $t_1, \cdots, t_n$ are n indeterminates. Let L be the algebra of linear exponentials of $P[t_1, \cdots, t_n]$ as defined in Section 1 of [3]. This means that L is the group ring with respect to $P[t_1, \cdots, t_n]$ of the group H of linear exponentials. A linear exponential is a symbol $\exp(\alpha_1 t_1 + \cdots + \alpha_n t_n)$ where $\alpha_i \in P$ and where the group H is defined by $\exp(\sum_{j=1}^{n} \alpha_j t_j) \exp(\sum_{j=1}^{n} \beta_j t_j) = \exp(\sum_{j=1}^{n} (\alpha_j + \beta_j) t_j)$. Let U be the m-dimensional column vector space over L as scalar domain. This means that U consists of the column vectors whose m components are elements of L. Since $\exp(\omega)$, where ω is the zero element of P, is the unit element of H, U contains as a submodule the m-dimensional column vector space W over $P[t_1, \cdots t_n]$ as scalar domain. The partial differential operators

$$\pi(D_1, \cdots, D_n) = \Sigma_{j_1 \cdots j_n} \gamma_{j_1 \cdots j_n} D_1^{j_1} \cdots D_n^{j_n}$$

where $\gamma_{j_1 \cdots j_n} \in P$ are defined as in [3] for the elements $\xi \in L$. Hence, $\pi(D_1, \cdots, D_n)(\xi)$ is computed by treating $D_1^{j_1} \cdots D_n^{j_n}$ as the differential operator $\partial^{j_1 + \cdots + j_n} / \partial t_1^{j_1} \cdots \partial t_n^{j_n}$ of analysis, the elements of P as constants and the symbol $\exp(\sum_{j=1}^{n} \alpha_j t_j)$ as the exponential $e^{\alpha_1 t_1 + \cdots + \alpha_n t_n}$ of analysis. Since the operators $\pi(D_1, \cdots, D_n)$ can be added and multiplied as ordinary polynomials, these operators form a domain $P[D_1, \cdots, D_n]$ which is isomorphic with the polynomial domain $P[x_1, \cdots, x_n]$. Under this isomorphism, to every operator $\pi(D_1, \cdots, D_n)$ there corresponds a polynomial $\pi(x_1, \cdots, x_n)$ which is called the *auxiliary polynomial* of $\pi(D_1, \cdots, D_n)$. We now consider the row vector operators $v(D_1, \cdots, D_n)$ which are row vectors whose m components are elements of $P[D_1, \cdots, D_n]$. If the components of $v(D_1, \cdots, D_n)$ are $\pi_1, \cdots, \pi_m$ and if $u \in U$ has the components $\xi_1, \cdots, \xi_m$, we define $v(D_1, \cdots, D_n)(u) = \sum_{j=1}^{m} \pi_j(D_1, \cdots, D_n)(\xi_j)$. Hence, the operators $v(D_1, \cdots, D_n)$ transform vectors of U in elements of L, they can be added

and subtracted as ordinary row vectors with m components, and can be multiplied by operators of $P[D_1, \cdots, D_n]$ in the usual way of scalar multiplication. Hence, the operators $v(D_1, \cdots, D_n)$ form a vector space $V[D_1, \cdots, D_n]$ over $P[D_1, \cdots, D_n]$ as scalar domain which is isomorphic with the m-dimensional row vector space V over $P[x_1, \cdots, x_n]$ as scalar domain. Under this isomorphism, to every operator vector $v(D_1, \cdots, D_n) \in V[D_1, \cdots, D_n]$ there corresponds a vector $v \in V$ which is called the *auxiliary vector* of $v(D_1, \cdots, D_n)$. A system S of differential operators of $V[D_1, \cdots, D_n]$ which is closed under vector subtraction and scalar multiplication is called an *operator module.* The auxiliary vectors of an operator module clearly form a module of V which is called the *auxiliary module* of S.

In the remainder of this section, S is a fixed operator module, M is its auxiliary module and $\mathfrak{p}$ is the prime ideal $\mathfrak{p} = (x_1, \cdots, x_n)$ of $P[x_1, \cdots, x_n]$. As in [1], Sections **2. 3** and **2. 5**, the module $M(\mathfrak{p})$ is defined as the isolated component of M all of whose associated primes are contained in $\mathfrak{p}$, and the module $M'(\mathfrak{p})$ as the isolated component of M all of whose associated primes are *properly* contained in $\mathfrak{p}$. A vector $w \in W$, i. e. a column vector whose m components are polynomials of $P[t_1, \cdots, t_n]$, is called a *modal column* of S if $v(D_1, \cdots, D_n)(w) = 0$ for all $v(D_1, \cdots, D_n) \in S$. The term modal column is taken from the theory of systems of ordinary differential equations. (See [4], p. 179.) The theory of systems of partial differential equations of the next section is based on the following theory of modal columns.

Let S_k be the sub-operator module of S whose auxiliary module is $M \cap \mathfrak{p}^k V$, i. e., S_k consists of those operators of S whose components have no derivatives of order less than k. (See [1], Section **2. 1** for the definition of a product such as $\mathfrak{p}^k V$.) A column vector $w \in W$ is called *homogeneous of degree k* or a *vector form of degree k* if not all of its m components are zero and if every non-zero component of w is a homogeneous polynomial of degree k of $P[t_1, \cdots, t_n]$. The degree of an arbitrary vector of W is of course the highest degree among its components. A vector w of degree k can be written uniquely as $w = \sum_{j=0}^{k} w_j$ where w_j is a vector form of degree j or is the zero vector and where $w_k \neq 0$.

LEMMA 1. 11. *Let $w = \sum_{j=0}^{k} w_j$ be the decomposition of the modal column w of degree k of S in homogeneous vector forms. Then, w_k is a modal column of S_k. Conversely, if w_k is a homogeneous modal column of degree k of S_k, we can find vector forms $w_{k-1}, \cdots, w_0$, where w_j has degree j or is the zero vector, such that $\sum_{j=0}^{k} w_j$ is a modal column of S.*

Proof. Since the vector operators $v_\alpha(D_1, \cdots, D_n) \in S_\alpha$ can be considered as linear functions on the vector space of vector forms w_α of degree α of W as $v_\alpha\ (D_1, \cdots, D_n)(w_\alpha) \in P$, the above lemma is proved in exactly the same way as Lemma 1.1 of [3].

Let $S(\mathfrak{p})$ be the operator module of $V[D_1, \cdots, D_n]$ whose auxiliary module is $M(\mathfrak{p})$.

LEMMA 1.12. *The vector w is a modal column of S if and only if w is a modal column of $S(\mathfrak{p})$.*

The proof of this lemma is the same as the proof of the more general Lemma 1.21 and hence is omitted here.

Now consider the sequence (1) $M(\mathfrak{p}) \subset M(\mathfrak{p}) : \mathfrak{p} \subset M(\mathfrak{p}) : \mathfrak{p}^2 \subset \cdots$ where the symbol $\subset$, *as everywhere else in this paper,* denotes *proper* inclusion (see [1] Section **2.1** for the definition of quotients such as $M(\mathfrak{p}) : \mathfrak{p}^k$). Since the ascending chain condition holds in V (see [6], Section 99), the last term of this sequence is well defined and is the isolated component of $M(\mathfrak{p})$ whose associated primes do not contain $\mathfrak{p}$ (see [5] Theorem 13). It follows that, since $M(\mathfrak{p}) = M'(\mathfrak{p}) \cap Q$ where Q is a primary component of M (see [1] Section **2.2**) whose associated prime is $\mathfrak{p}$ (or $M(\mathfrak{p}) = M'(\mathfrak{p})$ if $\mathfrak{p}$ is not an associated prime of M) and since the associated primes of $M'(\mathfrak{p})$ are properly contained in $\mathfrak{p}$, the last term of sequence (1) is exactly $M'(\mathfrak{p})$. Furthermore, if ρ is the exponent of the fundamental ideal of Q (see [1] Section **2.1** for the definition of fundamental ideal), $M'(\mathfrak{p}) \cap \mathfrak{p}^\rho V \subseteq M(\mathfrak{p})$. The *modal exponent* δ of the operator module S is defined as the smallest integer such that $M'(\mathfrak{p}) \cap \mathfrak{p}^\delta V \subseteq M(\mathfrak{p})$.

We assert that, for any k, the factor module $M'(\mathfrak{p}) \cap \mathfrak{p}^k V / M'(\mathfrak{p}) \cap \mathfrak{p}^{k+1} V$ has finite P-rank, say σ_k. (P-rank means rank with respect to P. See [1] Section **1, 1.**) The P-rank of the factor module $\mathfrak{p}^k V / \mathfrak{p}^{k+1} V$ is clearly finite, namely precisely $m(n+k-1)!/k!(n-1)!$ Consequently, the maximal number of vectors of $M'(\mathfrak{p}) \cap \mathfrak{p}^k V$ which is P-linearly independent mod. $\mathfrak{p}^{k+1} V$ is finite and this number is the P-rank of $M'(\mathfrak{p}) \cap \mathfrak{p}^k V / M'(\mathfrak{p}) \cap \mathfrak{p}^{k+1} V$. In the same way, the P-rank of $M(\mathfrak{p}) \cap \mathfrak{p}^k V / M(\mathfrak{p}) \cap \mathfrak{p}^{k+1} V$ is finite, say τ_k.

LEMMA 1.13. *Always $\sigma_k \geq \tau_k$ and $\lambda_k = \sigma_k - \tau_k$ is always the P-rank of the factor module $(M(\mathfrak{p}), M'(\mathfrak{p}) \cap \mathfrak{p}^k V) / (M(\mathfrak{p}), M'(\mathfrak{p}) \cap \mathfrak{p}^{k+1} V)$. The factor module $M'(\mathfrak{p})/M(\mathfrak{p})$ has finite P-rank $\lambda = \sum_{k=0}^{\delta-1} \lambda_k$, where δ is the modal exponent of S. If $k \geq \delta$, $\lambda_k = 0$ and hence $\sigma_k = \tau_k$.*

Proof. For the notion of sum of modules as $(M(\mathfrak{p}), M'(\mathfrak{p}) \cap \mathfrak{p}^k V)$, which

is the sum of the module $M(\mathfrak{p})$ and the module $M'(\mathfrak{p}) \cap \mathfrak{p}^k V$, see [1] Section 2.1. From the sequence

$$(1) \qquad M'(\mathfrak{p}) \cap \mathfrak{p}^\delta V \subset M'(\mathfrak{p}) \cap \mathfrak{p}^{\delta-1} V \subset \cdots \subset M'(\mathfrak{p}) \cap \mathfrak{p} V \subset M'(\mathfrak{p})$$

and the fact that $M'(\mathfrak{p}) \cap \mathfrak{p}^\delta V \subseteq M(\mathfrak{p})$, it follows that the maximal number of vectors of the module $M'(\mathfrak{p}) \cap \mathfrak{p}^k V$ which are P-linearly independent mod. $M(\mathfrak{p})$ is finite, i. e., the P-rank of the module $M'(\mathfrak{p}) \cap \mathfrak{p}^k V / M(\mathfrak{p}) \cap \mathfrak{p}^k V$ is finite, say μ_k. Furthermore $(M(\mathfrak{p}), M'(\mathfrak{p}) \cap \mathfrak{p}^k V)/(M(\mathfrak{p}), M'(\mathfrak{p}) \cap \mathfrak{p}^{k+1} V)$ is isomorphic to M_1/M_2 where $M_1 = (M(\mathfrak{p}), M'(\mathfrak{p}) \cap \mathfrak{p}^k V)/M(\mathfrak{p})$ and $M_2 = (M(\mathfrak{p}), M'(\mathfrak{p}) \cap \mathfrak{p}^{k+1} V)/M(\mathfrak{p})$. Since M_1 is isomorphic to $M'(\mathfrak{p}) \cap \mathfrak{p}^k V / M(\mathfrak{p}) \cap \mathfrak{p}^k V$ and M_2 is isomorphic to $M'(\mathfrak{p}) \cap \mathfrak{p}^{k+1} V / M(\mathfrak{p}) \cap \mathfrak{p}^{k+1} V$, the rank λ_k of M_1/M_2 satisfies $\lambda_k = \mu_k - \mu_{k+1}$. It is seen immediately from the sequence (1) and the corresponding sequence for $M(\mathfrak{p})$ that the rank of $M'(\mathfrak{p}) \cap \mathfrak{p}^k V / M(\mathfrak{p}) \cap \mathfrak{p}^{k+1} V$ is equal to both $\mu_k + \tau_k$ and $\mu_{k+1} + \sigma_k$ and hence that $\lambda_k = \sigma_k - \tau_k$. The sequence $M(\mathfrak{p}) = (M(\mathfrak{p}), M'(\mathfrak{p}) \cap \mathfrak{p}^\delta V) \subset (M(\mathfrak{p}), M'(\mathfrak{p}) \cap \mathfrak{p}^{\delta-1} V) \subset \cdots \subset (M(\mathfrak{p}), M'(\mathfrak{p}) \cap \mathfrak{p} V) \subset M'(\mathfrak{p})$ then shows that the P-rank of $M'(\mathfrak{p})/M(\mathfrak{p})$ is finite and equal to $\lambda = \sum_{k=0}^{\delta-1} \lambda_k$. Finally, if $k \geq \delta$, $M'(\mathfrak{p}) \cap \mathfrak{p}^k V \subseteq M'(\mathfrak{p}) \cap \mathfrak{p}^\delta V \subseteq M(\mathfrak{p})$ and hence $(M(\mathfrak{p}), M'(\mathfrak{p}) \cap \mathfrak{p}^k V) = M(\mathfrak{p})$. Consequently, the P-rank of $(M(\mathfrak{p}), M'(\mathfrak{p}) \cap \mathfrak{p}^k V)/(M(\mathfrak{p}), M'(\mathfrak{p}) \cap \mathfrak{p}^{k+1} V)$ is then zero, i. e., $\lambda_k = 0$.

The *modal rank* λ of the operator module S is defined as the P-rank if $M'(\mathfrak{p})/M(\mathfrak{p})$. Let $S'(\mathfrak{p})$ be the operator module of $V[D_1, \cdots, D_n]$ whose auxiliary module is $M'(\mathfrak{p})$. Since $S(\mathfrak{p}) \subseteq S'(\mathfrak{p})$, the modal columns of $S'(\mathfrak{p})$ are also modal columns of $S(\mathfrak{p})$, i. e., of S. A modal column of S which is at the same time a modal column of $S'(\mathfrak{p})$ is called a *trivial modal column* of S; otherwise the modal column is said to be *non-trivial*. These definitions of trivial and non-trivial will be justified by Lemma 1.22 and Remark 1.21.

For each k, let $v_{k1}, \cdots, v_{k\sigma_k}$ be σ_k vectors of $M'(\mathfrak{p}) \cap \mathfrak{p}^k V$ which are P-linearly independent mod. $M'(\mathfrak{p}) \cap \mathfrak{p}^{k+1} V$ and let $u_{k1}, \cdots, u_{k\tau_k}$ be τ_k vectors of $M(\mathfrak{p}) \cap p^k V$ which are P-linearly independent mod. $M(\mathfrak{p}) \cap \mathfrak{p}^{k+1} V$. Since $M(\mathfrak{p}) \cap \mathfrak{p}^k V \subseteq M'(\mathfrak{p}) \cap \mathfrak{p}^k V$ we can find, for each k, a $\tau_k \times \sigma_k$ matrix (α_{ij}) where $\alpha_{ij} \in P$, $i = 1, \cdots, \tau_k$, $j = 1, \cdots, \sigma_k$, such that

$$(1) \qquad \begin{pmatrix} u_{k1} \\ \cdot \\ \cdot \\ \cdot \\ u_{k\tau_k} \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1\sigma_k} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \alpha_{\tau_k 1} & \cdots & \alpha_{\tau_k \sigma_k} \end{pmatrix} \begin{pmatrix} v_{k1} \\ \cdot \\ \cdot \\ \cdot \\ v_{k\sigma_k} \end{pmatrix} \quad \text{mod } M'(\mathfrak{p}) \cap \mathfrak{p}^{k+1} V.$$

The α_{ij} depend of course on k but the lower index k will be omitted for the α_{ij}'s. Each vector v_{kj} is the auxiliary vector of an operator $v_{kj}(D_1, \cdots, D_n) \in S'(\mathfrak{p})$. Furthermore, the vectors v_{kj} $(j = 1, \cdots, \sigma_k)$ are P-linearly independent mod. $\mathfrak{p}^{k+1}V$ since, otherwise, we could find constants $\beta_j \in P$ such that $\sum_{j=1}^{\sigma_k} \beta_j v_{kj} \in M'(\mathfrak{p}) \cap \mathfrak{p}^{k+1}V$. It follows easily from this and from the fact that $v_{kj}(D_1, \cdots, D_n)$ has $m(n+k-1)!/k!(n-1)!$ derivatives of order k, that we can find $(m(n+k-1)!/k!(n-1)!) - \sigma_k = \mu_k$ P-linearly independent vector forms of degree k, $s_{k1}, \cdots, s_{k\mu_k}$ where $s_{kj} \in W$, such that $v_{kj}(D_1, \cdots, D_n)(s_{ki}) = 0$ for $j = 1, \cdots, \sigma_k$ and $i = 1, \cdots, \mu_k$. Since each vector of $M'(\mathfrak{p}) \cap \mathfrak{p}^k V$ is P-linearly dependent on $v_{k1}, \cdots, v_{k\sigma_k}$ mod. $M'(\mathfrak{p}) \cap \mathfrak{p}^{k+1}V$, the vector forms $s_{k1}, \cdots, s_{k\mu_k}$ are homogeneous modal columns of degree k of $S'_k(\mathfrak{p})$. According to Lemma 1.11 we can find μ_k modal columns $a_{k1}, \cdots, a_{k\mu_k}$ of degree k of $S'(\mathfrak{p})$, i.e., trivial modal columns of S, where s_{kj} is the vector form of degree k in the decomposition of a_{kj} in vector forms. Each vector u_{kj} is the auxiliary vector of an operator $u_{kj}(D_1, \cdots, D_n) \in S_k(\mathfrak{p})$ for $j = 1, \cdots, \tau_k$. Furthermore, the rows of the matrix (α_{ij}) are P-linearly independent since otherwise we could find constants $\beta_j \in P$ such that $\sum_{j=1}^{\tau_k} \beta_j u_{kj} \in M(\mathfrak{p}) \cap \mathfrak{p}^{k+1}V$; consequently, λ_k (see Lemma 1.12) is the rank of the null space of (α_{ij}). It follows easily, from this and from the fact that $v_{k1}, \cdots, v_{k\sigma_k}$ are P-linearly independent mod. $\mathfrak{p}^{k+1}V$, that we can find λ_k P-linearly independent vector forms of degree k, $e_{k1}, \cdots, e_{k\lambda_k}$, where $e_{kj} \in W$, such that $u_{kj}(D_1, \cdots, D_n)(e_{ki}) = 0$ for $j = 1, \cdots, \tau_k$, $i = 1, \cdots, \lambda_k$ *and where no* e_{ki} *satisfies all the* σ_k *operators* $v_{kj}(D_1, \cdots, D_n)$ $(j = 1, \cdots, \sigma_k)$. Since every vector of $M(\mathfrak{p}) \cap \mathfrak{p}^k V$ is P-linearly dependent on $u_{k1}, \cdots, u_{k\tau_k}$ mod. $M(\mathfrak{p}) \cap \mathfrak{p}^{k+1}V$ each e_{ki} is a homogeneous modal column of degree k of $S_k(\mathfrak{p})$. According to Lemma 1.11 we can find λ_k modal columns $f_{k1}, \cdots, f_{k\lambda_k}$ of degree k of $S(\mathfrak{p})$, where e_{kj} is the vector form of degree k which occurs in the decomposition of f_{kj} in vector forms. Since e_{kj} is not a modal column of $S'_k(\mathfrak{p})$, it follows from Lemma 1.1 that f_{kj} is not a modal column of $S'(\mathfrak{p})$ and hence that $f_{k1}, \cdots, f_{k\lambda_k}$ are λ_k non-trivial modal columns of $S(\mathfrak{p})$ and hence of S.

LEMMA 1.14. *The above vectors* $f_{kj} \in W$ *for* $0 \leq k \leq \delta - 1$ *and* $1 \leq j \leq \lambda_k$ *are* λ *non-trivial* P*-linearly independent modal columns of* S, *where* δ *is the modal exponent of* S, λ *the modal rank of* S, *and* k *is the degree of* f_{kj}. *An arbitrary modal column* $w \in W$ *of* S *is* P*-linearly dependent on the* λ *columns above and on a trivial modal column. If the degree of* w

is $k \leq \delta - 1$, w is P-linearly dependent on the $\sum_{q=0}^{k} \lambda_q$ modal columns f_{qj} for $0 \leqq q \leqq k$ and $1 \leqq j \leqq \lambda_q$ and on a trivial modal column.

Proof. We already know that the f_{kj}'s are non-trivial modal columns of S, that k is the degree of f_{kj} and that the number of f_{kj}'s, mentioned in Lemma 1.14, is $\sum_{k=0}^{\delta-1} \lambda_k$ which is the modal rank λ of S. Since e_{kj} is the vector form of degree k which occurs in the decomposition of f_{kj} in vector forms, the P-linear independence of the e_{kj}'s implies the P-linear independence of the λf_{kj}'s of Lemma 1.14. A vector form $w_k \varepsilon W$ of degree k is a modal column of $S_k(\mathfrak{p})$ if and only if $u_{kj}(D_1, \cdots, D_n)(w_k) = 0$ for $1 \leq j \leq \tau_k$. It follows from this fact and equation (1) that an arbitrary homogeneous modal column w_k of degree k of $S_k(\mathfrak{p})$ is P-linearly dependent on the above $e_{k1}, \cdots, e_{k\lambda_k}, s_{k1}, \cdots, s_{k\mu_k}$. Let $w = \sum_{j=0}^{k} w_j$ be the decomposition of an arbitrary modal column w of degree k of S and hence of $S(\mathfrak{p})$ in vector forms. According to Lemma 1.11, w_k is then a homogeneous modal column of degree k of $S_k(\mathfrak{p})$ and hence

$$w_k = \sum_{j=1}^{\lambda_k} \alpha_j e_{kj} + \sum_{j=1}^{\mu_k} \beta_j s_{kj}. \tag{2}$$

If $k \geq \delta$, $\lambda_k = 0$ according to Lemma 1.13 and hence, in that case, $w_k = \sum_{j=1}^{\mu_k} \beta_j s_{kj}$. Consider then the vector $g = f - \sum_{j=1}^{\mu_k} \beta_j a_{kj}$ where the a_{kj}'s have the same meaning as before and where hence $\sum_{j=1}^{\mu_k} \beta_j a_{kj}$ is a trivial modal column of S. Since g has degree at most $k-1$, it follows that f is the sum of a modal column of degree at most $k-1$ and a trivial modal column. Repetition of this process with g until the degree has become at most $\delta - 1$ shows that *every modal column of S is the sum of a modal column of degree at most $\delta - 1$ and a trivial modal column.* If $k \leq \delta - 1$, equation (2) holds and we consider $g = f - \sum_{j=1}^{\lambda_k} \alpha_j f_{kj} - \sum_{j=1}^{\mu_k} \beta_j a_{kj}$, where g is a modal column of S of degree at most $k-1$. Hence, f is then the sum of a modal column of degree at most $k-1$ of S, a modal column which is P-linearly dependent on $f_{k1}, \cdots, f_{k\lambda_k}$, and a trivial modal column. Repetition of this process with g until the degree has become zero shows that f is the sum of a modal column h of degree zero, a modal column which is P-linearly dependent on f_{qj} for $1 \leq q \leq k$ and $1 \leq j \leq \lambda_k$, and a trivial modal column. Since the components of h are all elements of P, h either is the zero vector or is homogeneous and of degree zero and consequently P-linearly dependent on

$e_{01}, \cdots, e_{0\lambda_0}, s_{01}, \cdots, s_{0\lambda_0}$. Since $e_{0j} = f_{0j}$ for $1 \leq j \leq \lambda_0$, the lemma is proved.

1. 2. Systems of partial differential equations. This section is an extension of Section 2 of [3] to m dependent variables and of Section **1. 4** of [1] to n independent variables. A system of homogeneous, partial, linear differential equations with constant coefficients in m dependent variables and n independent variables is determined by a system S of operators of $V[D_1, \cdots, D_n]$. (See **1. 1.**) A solution of S is defined as an element $z \in U$ such that $v(D_1, \cdots, D_n)(z) = 0$ for all $v(D_1, \cdots, D_n) \in S$. Here, U is the m-dimensional column vector space over M as scalar domain, where M is the algebra of linear exponentials of $K[t_1, \cdots, t_n]$ and K is any extension field of P (see **1. 1**). From the isomorphism $V[D_1, \cdots, D_n] \cong V$, where V is the m-dimensional row vector space over $P[x_1, \cdots, x_n]$ as scalar domain (see **1. 1**), it follows that S generates an operator module. Since it is clear that this operator module and S have the same solutions, it is assumed hereafter that $S \subseteq V[D_1, \cdots, D_n]$ is a fixed operator module and $M \subseteq V$ is its auxiliary module.

Let $w \exp(\xi t)$, be a vector of U, where (ξt) denotes the linear form $\sum_{j=1}^{n} \xi_j t_j$, $\xi_j \in K$, and where w is a vector of the m-dimensional column vector space W over $K[t_1, \cdots, t_n]$ as scalar domain. If $v(D_1, \cdots, D_n) \in V[D_1, \cdots, D_n]$, then $v(D_1, \cdots, D_n)\ (w \exp(\xi t))$ either is the zero element of M or is equal to $h \exp(\xi t)$ where $h \neq 0$ and $h \in P[t_1, \cdots, t_n]$. Observe that, if $\pi(D_1, \cdots, D_n) \in P[D_1, \cdots, D_n]$ and h is a non-zero polynomial of $P[t_1, \cdots, t_n]$, $\pi(D_1, \cdots, D_n)\ h \exp(\xi t)$ cannot be zero unless $\pi(\xi_1, \cdots, \xi_n) = 0$, where $\pi(\xi_1, \cdots, \xi_n)$ is the value of the auxiliary polynomial $\pi \in P[x_1, \cdots, x_n]$ of $\pi(D_1, \cdots, D_n)$, for $x_j = \xi_j (1 \leq j \leq n)$. This follows from the fact that

$$\pi(D_1, \cdots, D_n) h \exp(\xi t) = \pi(\xi_1, \cdots, \xi_n)\, h \exp(\xi t) + g \exp(\xi t)$$

where $g \in P[t_1, \cdots, t_n]$ is a polynomial whose degree is less than the degree of h.

In order to investigate the relationship of $w \exp(\xi t)$ to S, let $\mathfrak{p} \subseteq P[x_1, \cdots, x_n]$ be the prime ideal which consists of the polynomials which vanish for $x_j = \xi_j$, $1 \leq j \leq n$. Again $M(\mathfrak{p})$ denotes the isolated component of M whose associated primes are contained in $\mathfrak{p}$ and $M'(\mathfrak{p})$ the isolated component of M whose associated primes are properly contained in $\mathfrak{p}$. $S(\mathfrak{p})$ and $S'(\mathfrak{p})$ are the operator modules of $V[D_1, \cdots, D_n]$ which have respectively $M(\mathfrak{p})$ and $M'(\mathfrak{p})$ as auxiliary modules.

LEMMA 1.21. *The vector* $w \exp(\xi t)$, *where* $w \in W$, *is a solution of* S *if and only if* $w \exp(\xi t)$ *is a solution of* $S(\mathfrak{p})$.

Proof. Since $S \subseteq S(\mathfrak{p})$, every solution of $S(\mathfrak{p})$ is a solution of S. Suppose that $w \exp(\xi t)$ is a solution of S. If $M = Q_1 \cap \cdots \cap Q_h$ is a Noether Decomposition of M and $\mathfrak{p}_j$ is the radical of Q_j for $1 \leq j \leq h$, then $M(\mathfrak{p}) = Q_1 \cap \cdots \cap Q_m$ if $\mathfrak{p}_j \subseteq \mathfrak{p}$ for $1 \leq j \leq m$ and $\mathfrak{p}_j \not\subseteq \mathfrak{p}$ for $m < j \leq h$. Consequently, if ρ is the maximum of the exponents of the fundamental ideals of $Q_{m+1}, \cdots, Q_h$, then $(\Pi^h_{j=m+1}\mathfrak{p}_j)^\rho M(\mathfrak{p}) \subseteq M$. Since $\mathfrak{p}_j \not\subseteq \mathfrak{p}$ for $m < j \leq h$, we can find $h - m$ polynomials $\pi_j \in P[x_1, \cdots, x_n]$ such that $\pi_j \in \mathfrak{p}_j$ and $\pi_j(\xi_1, \cdots, \xi_n) \neq 0$ for $m < j \leq h$. The polynomial $\pi = (\Pi^h_{j=m+1}\pi_j)^\rho$ then has the properties $\pi \in (\Pi^h_{j=m+1}\mathfrak{p}_j)^\rho$ and $\pi(\xi_1, \cdots, \xi_n) \neq 0$. Let $\pi(D_1, \cdots, D_n)$ be the operator of $V[D_1, \cdots, D_n]$ whose auxiliary polynomial is π. If $w \exp(\xi t)$ is not a solution of $S(\mathfrak{p})$, there exists an operator $v(D_1, \cdots, D_n) \in S(\mathfrak{p})$ such that $v(D_1, \cdots, D_n)\ w \exp(\xi t) = g \exp(\xi t)$ where $g \in W$ and $g \neq 0$. The auxiliary operator v of $v(D_1, \cdots, D_n)$ is a vector of $M(\mathfrak{p})$ and hence $\pi v \in M$ which implies $\pi(D_1, \cdots, D_n) v(D_1, \cdots, D_n) \in S$. However, $\pi(D_1, \cdots, D_n)\ v(D_1, \cdots, D_n)\ (w \exp(\xi t)) = \pi(D_1, \cdots, D_n)\ (g \exp(\xi t))$ is not zero since $g \neq 0$ and $\pi(\xi_1, \cdots, \xi_n) \neq 0$ and hence $w \exp(\xi t)$ would not be a solution of S which is against the hypothesis. Hence, $w \exp(\xi t)$ is a solution of $S(\mathfrak{p})$ and the lemma is proved.

Since $S(\mathfrak{p}) \subseteq S'(\mathfrak{p})$, every solution of $S'(\mathfrak{p})$ is a solution of $S(\mathfrak{p})$, i. e. of S.

DEFINITION 1.21. *If a solution* $w \exp(\xi t)$ *of* S, *where* $w \in W$, *is also a solution of* $S'(\mathfrak{p})$, *the solution is said to be a trivial solution of* S. *Otherwise,* $w \exp(\xi t)$ *is called a non-trivial solution of* S.

Remark 1.21. This notion of triviality is an immediate extension of the notion of triviality used in [1], Section **1.41** and **1.61**. If $n = 1$, $M'(\mathfrak{p})$ is always the closure of M, since then the non-zero prime ideals are always maximal.

We know from [1] that, if $\mathfrak{p}$ is not an associated prime ideal of M, $M(\mathfrak{p}) = M'(\mathfrak{p})$. Hence then $S(\mathfrak{p}) = S'(p)$ and all solutions $w \exp(\xi t)$ are trivial. In order to discuss the non-trivial solutions of S, the following assumptions are made for the remainder of this section: the ideal $\mathfrak{p} \subseteq P[x_1, \cdots, x_n]$ is a fixed, $(n\text{-}i)$-dimensional associated prime ideal of M; $\xi_1, \cdots, \xi_n$ is a general point of $\mathfrak{p}$ where ξ_j corresponds to the restclass of $x_j \bmod \mathfrak{p}$ for $1 \leq j \leq n$; the variables $x_1, \cdots, x_n$ are ordered in such a way that $\xi_{t+1}, \cdots, \xi_n$ are algebraically independent with respect to P and each ξ_j

for $1 \leq j \leq i$, is algebraic with respect to the field $\bar{P} = P(\xi_{i+1}, \cdots, \xi_n)$; the vectors $\xi_1^{(j)}, \cdots, \xi_i^{(j)}$ for $1 \leqq j \leqq c$, where $\xi_s^{(j)}$ is an element of the algebraic closure of $\bar{P}$, are the vectors which are conjugated to $\xi_1, \cdots, \xi_i = \xi_1^{(1)}, \cdots, \xi_i^{(1)}$ with respect to P; P^{**} is the field which is obtained from P by adjunction to $\bar{P}$ of all $\xi_s^{(j)}$ for $1 \leq j \leq c$, $1 \leq s \leq i$. We want to discuss those solutions $w^{**} \exp(\xi t)$ of S where (ξt) is $\xi_1 t_1 + \cdots + \xi_n t_n$ and where the m components of the column vector w^{**} are polynomials of $P^{**}[t_1, \cdots, t_i]$. The solution $w^{**} \exp(\xi t)$ is considered as a vector of the m-dimensional column vector space over the algebra of linear exponentials of $P^{**}[t_1, \cdots, t_n]$ as scalar domain.

Remark 1.22. It may seem unnatural to allow only the variables $t_1, \cdots, t_i$ in the components of w^{**}, i.e., the variables which correspond to the algebraic coordinates of $\xi_1, \cdots, \xi_n$. The justification is given by Section 3 of [3], where it is shown that, in the case of one dependent variable, this restriction of w^{**} arises in a natural way from the consideration of the initial conditions which may be imposed on the solutions of S.

If in each component of a vector $v \in M(\mathfrak{p})$ we carry out the substitution $(\Xi) x_j = \xi_j$, $j = i+1, \cdots, n$, this vector becomes a vector $\bar{v} \in \bar{V}$ where $\bar{V}$ is the m-dimensional row vector space over $\bar{P}[x_1, \cdots, x_i]$ as scalar domain. Let $\overline{M(\mathfrak{p})}$ denote the module of $\bar{V}$ which is generated by all vectors of $M(\mathfrak{p})$ after the substitution (Ξ) has been carried out. Since $\bar{P} \subseteq P^{**}$, the vectors of $\overline{M(\mathfrak{p})}$ can also be considered as vectors of the m-dimensional row vector space V^{**} over $P^{**}[x_1, \cdots, x_i]$ as scalar domain. Let $M^{**}(\mathfrak{p})$ denote the module of V^{**} which is generated by the vectors of $\overline{M(\mathfrak{p})}$. Each component of each vector $v^{**} \in M^{**}(\mathfrak{p})$ can be expanded with respect to the products $(x_1 - \xi_1)^{j_1} \cdots (x_i - \xi_i)^{j_i}$, i.e., if v^{**} has the components $\pi^{**}{}_1, \cdots, \pi^{**}{}_m$, where $\pi^{**}{}_k \in P^{**}[x_1, \cdots x_i]$ for $1 \leq k \leq m$, then

$$\pi^{**}{}_k = \Sigma_{j_1 \cdots j_i} a^{**}{}_{k j_1 \cdots j_i} (x_1 - \xi_1)^{j_1} \cdots (x_i - \xi_i)^{j_i}$$

where $a^{**}{}_{k j_1 \cdots j_i} \in P^{**}$. If in each such expanded component we carry out the substitution $(D)(x_j - \xi_j) = D_j$, $j = 1, \cdots, i$, we obtain an operator module $S^{**}{}_X(\mathfrak{p})$ of $V^{**}[D_1, \cdots, D_i]$ over $P^{**}[D_1, \cdots, D_i]$ as scalar domain. Observe that the auxiliary module of $S^{**}{}_X(\mathfrak{p})$ is not $M^{**}(\mathfrak{p})$ but is the module $M^{**}{}_X(\mathfrak{p})$ which arises from the expanded components of the vectors of $M^{**}(\mathfrak{p})$ by the substitution (X) $x_j - \xi_j = X_j$, $j = 1, \cdots, i$, where $X_1, \cdots, X_i$ are i determinates.

LEMMA 1.22. *The vector $w^{**} \exp(\xi t)$, where, as above, the components of w^{**} are polynomials of $P^{**}[t_1, \cdots, t_i]$ and where $\xi_1, \cdots, \xi_n$ is a general*

point of the $(n-i)$-dimensional associated prime $\mathfrak{p}$ of M, is a solution of S if and only if w^{**} is a modal column of $S^{**}{}_x(\mathfrak{p})$. The solution $w^{**}\exp(\xi t)$ is a trivial solution of S if w^{**} is a trivial modal column of $S^{**}{}_x(\mathfrak{p})$ and is a non-trivial solution if w^{**} is a non-trivial modal column of $S^{**}{}_x(\mathfrak{p})$.

Proof. Since S and $S(\mathfrak{p})$ have the same solutions, let $v(D_1,\cdots,D_n)\in S(\mathfrak{p})$ and let $v\in M(\mathfrak{p})$ be the auxiliary vector of $v(D_1,\cdots,D_n)$. The vector v gives rise to an operator $v^{**}(D_1,\cdots,D_i)\in S^{**}{}_x(\mathfrak{p})$, where $v^{**}(D_1,\cdots,D_i)$ is obtained from v by the substitution $D\Xi$ (first Ξ and then D). These operators $v^{**}(D_1,\cdots,D_i)$ generate $S^{**}{}_x(\mathfrak{p})$ and furthermore $v(D_1,\cdots,D_n)(w^{**}\exp(\xi t))=\exp(\xi t)v^{**}(D_1,\cdots,D_i)(w^{*})$ which is proved by direct computation in exactly the same way as Lemma 2.2 of [3]. Since $\exp(\xi t)v^{**}(D_1,\cdots,D_i)(w^{**})=0$ if and only if $v^{**}(D_1,\cdots,D_i)(w^{*})=0$, we have proved that $w^{**}\exp(\xi t)$ is a solution of $S(\mathfrak{p})$, i. e., of S, if and only if w^{**} is a modal column of $S^{**}{}_x(\mathfrak{p})$. In the same way we show that $w^{**}\exp(\xi t)$ is a solution of $S'(\mathfrak{p})$, i. e., a trivial solution of S, if and only if w^{**} is a modal column of $S'^{**}{}_x(\mathfrak{p})$, where $S'^{**}{}_x(\mathfrak{p})$ arises from $S'(\mathfrak{p})$ in the same way as $S^{**}{}_x(\mathfrak{p})$ from $S(\mathfrak{p})$. Since the auxiliary module of $S'^{**}{}_x(\mathfrak{p})$ is $M'^{**}{}_x(\mathfrak{p})$, where $M'^{**}{}_x(\mathfrak{p})$ arises from $M'(\mathfrak{p})$ as $M^{**}{}_x(\mathfrak{p})$ arises from $M(\mathfrak{p})$, all there remains to be shown is that $M'^{**}{}_x(\mathfrak{p})$ is the isolated component of $M^{**}{}_x(\mathfrak{p})$ whose associated primes are properly contained in the ideal $\mathfrak{X}=(X_1,\cdots,X_i)$. For this purpose, we observe that $M(\mathfrak{p})=M'(\mathfrak{p})\cap Q$ where Q is any primary component of M whose radical is $\mathfrak{p}$ (see [1] Section **2.3**). This implies that $\overline{M(\mathfrak{p})}=\overline{M'(\mathfrak{p})}\cap\bar{Q}$ where the bar again indicates the substitution Ξ (see [6], Section 97, where the same statement is proved for ideals). Since P^{**} is an algebraic extension of $\bar{P}$, we conclude that then $M^{**}(\mathfrak{p})=M'^{**}(\mathfrak{p})\cap Q^{**}$. The module $\bar{Q}$ is a primary module which has $\bar{\mathfrak{p}}$ as radical where $\bar{\mathfrak{p}}$ is the zero dimensional prime ideal of $\bar{P}[x_1,\cdots,x_i]$ whose general point is $\xi_1,\cdots,\xi_i$ (see [6], Section 97, where the corresponding statement is proved for ideals; the proof can be immediately extended to modules). It follows that $Q^{**}=Q^{**}{}_1\cap\cdots\cap Q^{**}{}_c$ where $Q^{**}{}_j$ is a primary module of V^{**} which has $\mathfrak{p}^{**}(\xi^{(j)})=(x_1-\xi_1{}^{(j)},\cdots,x_i-\xi_i{}^{(j)})$ as radical for $1\leq j\leq c$ (see [7], Theorem 25; the proof of that theorem can immediately be extended to modules). Furthermore, from the fact that the associated primes of $M'(\mathfrak{p})$ are properly contained in $\mathfrak{p}$, it follows easily that the associated primes of $M'^{**}(\mathfrak{p})$ are properly contained in $\mathfrak{p}^{**}(\xi^{(j)})$, $1\leq j\leq c$. Consequently, $M^{**}(\mathfrak{p})=M'^{**}(\mathfrak{p})\cap Q^{**}{}_1\cap\cdots\cap Q^{**}{}_c$ and $M'^{**}(\mathfrak{p})$ is the isolated component of $M^{**}(\mathfrak{p})$ whose associated primes are properly contained in $\mathfrak{p}^{**}(\xi^{(1)})$. Since the substitution (X) establishes an isomorphism between the m-dimen-

sional row vector space V^{**} over $P^{**}[x_1, \cdots, x_i]$ and the m-dimensional row vector space $V^{**}{}_X$ over $P^{**}[X_1, \cdots, X_i]$ which transforms $M^{**}(\mathfrak{p})$ in $M^{**}{}_X(\mathfrak{p})$, $M'^{**}(\mathfrak{p})$ in $M'^{**}{}_X(\mathfrak{p})$ and $\mathfrak{p}^{**}(\xi^{(1)})$ in $\mathfrak{X}$, $M'^{**}{}_X(\mathfrak{p})$ is the isolated component of $M^{**}{}_X(\mathfrak{p})$ whose associated primes are properly contained in $\mathfrak{X}$.

The modal rank $\lambda(\mathfrak{p})$ (see **1.1**) of $S^{**}{}_X(\mathfrak{p})$ is the P^{**}-rank of the factor module $M'^{**}{}_X(\mathfrak{p})/M^{**}{}_X(\mathfrak{p})$. Because of the isomorphism (X) between $V^{**}{}_X$ and V^{**}, $\lambda(\mathfrak{p})$ is also the P^{**}-rank of $M'^{**}(\mathfrak{p})/M^{**}(\mathfrak{p})$. The modal exponent $\delta(\mathfrak{p})$ (see **1.1**) of $S^{**}{}_X(\mathfrak{p})$ is the smallest integer such that $M'^{**}{}_X(\mathfrak{p}) \cap \mathfrak{X}^{\delta(\mathfrak{p})} V^{**}{}_X \subseteq M^{**}{}_X(\mathfrak{p})$. If we call $\mathfrak{p}^{**}(\xi) = \mathfrak{p}^{**}(\xi^{(1)})$, the same isomorphism (X) shows that $\delta(\mathfrak{p})$ is the smallest integer such that $M'^{**}(\mathfrak{p}) \cap (\mathfrak{p}^{**}(\xi))^{\delta(\mathfrak{p})} V^{**}{}_X \subseteq M^{**}{}_X(\mathfrak{p})$.

DEFINITION 1.22. *Let M be a module of the m-dimensional row vector space V over $P[x_1, \cdots, x_n]$ as scalar domain. Let $\mathfrak{p}$ be an associated prime of M with general point $\xi_1, \cdots, \xi_n$ where all notations and assumptions are as above. Then, the $\mathfrak{p}$-modal rank $\lambda(\mathfrak{p})$ of M is defined as the P^{**}-rank of $M'^{**}(\mathfrak{p})/M^{**}(\mathfrak{p})$. The $\mathfrak{p}$-modal exponent $\delta(\mathfrak{p})$ of M is defined as the smallest integer such that $M'^{**}(\mathfrak{p}) \cap (\mathfrak{p}^{**}(\xi))^{\delta(\mathfrak{p})} V^{**} \subseteq M^{**}(\mathfrak{p})$.*

Let W^{**} denote the m-dimensional column vector space over $P^{**}[t_1, \cdots, t_i]$ as scalar domain. The following theorem is a corollary of Lemmas 1.14 and 1.22.

THEOREM 1.21. *There exist $\lambda(\mathfrak{p})$, P^{**}-linearly independent, non-trivial solutions $f^{**}{}_{kj} \exp(\xi t)$ of S. Here, $f^{**}{}_{kj} \in W^{**}$, the degree of $f^{**}{}_{kj}$ is k where $0 \leq k \leq \delta(\mathfrak{p}) - 1$, $1 \leq j \leq \lambda_k(\mathfrak{p})$ and hence $\sum_{k=0}^{\delta(\mathfrak{p})-1} \lambda_k(\mathfrak{p}) = \lambda(\mathfrak{p})$. An arbitrary solution $w^{**} \exp(\xi t)$ of S, where $w^{**} \in W^{**}$, is P^{**}-linearly dependent on the $\lambda(\mathfrak{p})$ solutions above and on a trivial solution of S. If the degree of w^{**} is $k \leq \delta(\mathfrak{p}) - 1$, $w^{**} \exp(\xi t)$ is P^{**}-linearly dependent on the $\sum_{q=0}^{k} \lambda_q(\mathfrak{p})$ solutions $f^{**}{}_{qj} \exp(\xi t)$ for $0 \leq q \leq k$, $1 \leq j \leq \lambda_q(\mathfrak{p})$, and on a trivial solution of S.*

Remark 1.23. If $m = 1$, Theorem 1.21 contains Theorem 2.1 of [3] as a corollary (in [3], $\delta(\mathfrak{p})$ is called the differential exponent instead of the $\mathfrak{p}$-modal exponent). If $n = 1$, Theorem 1.21 becomes Theorem 1.42 of [1]. Hence, the above theory of the exponential solutions of a system of partial differential equations and the theory of systems of linear algebraic equations of [1], Sections **2.6** and **2.7**, tell us what exponents of an $s \times m$ polynomial

matrix $A = (\alpha_{ij})$, where $\alpha_{ij} \in P[x_1, \cdots, x_n]$, we want to study. Namely, for each associated prime $\mathfrak{p}$ of the row space of A (i. e., the module generated by the rows of A), we are interested in $\lambda(\mathfrak{p})$ and $\delta(\mathfrak{p})$; and, for each associated prime $\mathfrak{p}$ of the column space of A, we are interested in the $\mathfrak{p}$-length $l(\mathfrak{p})$ (see [1] Section **2.6**). The algebraic investigation of these exponents is based on the theory of Part II of [1] and on the theory of the Hilbert characteristic function of Part II of the present paper.

II. Polynomial Modules.

2.1. Homogeneous modules and their Noether decompositions. The theory of homogeneous modules is needed for the theory of the Hilbert characteristic functions of modules. Let P be an arbitrary field, not necessarily of characteristic zero, let V be the m-dimensional row vector space over $P[x_1, \cdots, x_n]$ as scalar domain and let F be the m-dimensional row vector space over $P[x_0, x_1, \cdots, x_n]$ as scalar domain. A vector $v \in F$ is called a *vector form* or H-vector of degree $\partial(v)$, if not all components of v are zero and if each of the non-zero components is a homogeneous polynomial, called a "form," of degree ∂ of $P[x_0, x_1, \cdots, x_n]$. Elements of P are called constants and a constant vector is a vector whose components are elements of P. A non-zero constant vector is a vector form of degree zero and the zero vector is considered as a vector form of degree -1. The degree $\partial(v)$ of an *arbitrary* vector v is the highest degree of its components and v can be written as a unique sum of vector forms $v = \sum_{j=0}^{\partial(v)} w_j$ where w_j is a vector form of degree j or the zero vector and where $w_{\partial(v)} \neq 0$. The vector forms w_j are called the *homogeneous components of* v.

DEFINITION 2.11. *The module $H \subseteq F$ is called a homogeneous module or H-module if the homogeneous components of any $v \in H$ are themselves vectors of H.*

Since every module of F has a finite number of generators, a module of F is homogeneous if and only if it can be generated by a finite number of vector forms.

We shall now consider the Noether decomposition of an H-module. Let L and M be two H-modules and $\mathfrak{c}$ an H-ideal of $P[x_0, x_1, \cdots, x_n]$ (see [7] Section 3). The sum (L, M), the intersection $L \cap M$, the quotients $L : M$ and $M : \mathfrak{c}$, the product $\mathfrak{c}M$, the fundamental ideal $\mathfrak{f}(M)$ and the radical $\mathfrak{r}(M)$ are all defined in [1] as either modules of F or ideals of $P[x_0, \cdots, x_n]$. It

can be shown easily that these modules and ideals are always H-modules and H-ideals. The *homogeneous kernel* M_H of an arbitrary module $M \subseteq F$ is defined as the module which is generated by the vector forms of M. Clearly M_H is the largest H-module which is contained in M and, if M is homogeneous, $M = M_H$. If L and M are two modules of F, it can be easily shown that $(L \cap M)_H = L_H \cap M_H$.

LEMMA 2.11. *Let M be a module of F. Then, $(\mathfrak{f}(M))_H = \mathfrak{f}(M_H)$ and $(\mathfrak{r}(M))_H = \mathfrak{r}(M)_H$. If Q is a primary module of F with $\mathfrak{p}$ as radical, Q_H is primary and has $\mathfrak{p}_H$ as radical.*

Proof. Since $M_H \subseteq M$, $\mathfrak{f}(M_H) \subseteq \mathfrak{f}(M)$ and, since we already know that $\mathfrak{f}(M_H)$ is homogeneous, $\mathfrak{f}(M_H) \subseteq (\mathfrak{f}(M))_H$. Now, let the form π of $P[x_0, \cdots, x_n]$ be contained in $(\mathfrak{f}(M))_H$. Then $\pi \in \mathfrak{f}(M)$, which means that $\pi e_j \in M$ for $j = 1, \cdots, m$ where e_j is the vector of F whose only non-zero component is the j-th one which is equal to unity. Since πe_j is a vector form we conclude that $\pi e_j \in M_H$ for $j = 1, \cdots, m$, i.e. $\pi \in \mathfrak{f}(M_H)$. Consequently, all generators of $(\mathfrak{f}(M))_H$ are elements of $\mathfrak{f}(M_H)$ which proves that $(\mathfrak{f}(M))_H = \mathfrak{f}(M_H)$. Furthermore, $M_H \subseteq M$ implies, in the same way as for $\mathfrak{f}$, that $\mathfrak{r}(M_H) \subseteq (\mathfrak{r}(M))_H$. Let the form π of $P[x_0, \cdots, x_n]$ be an element of $(\mathfrak{r}(M))_H$. Then, $\pi \in \mathfrak{r}(M)$ and hence, for some integer ρ, $\pi^\rho \in \mathfrak{f}(M)$. Since π^ρ is a form we conclude that $\pi^\rho \in (\mathfrak{f}(M))_H$ and hence that $\pi^\rho \in \mathfrak{f}(M_H)$ from which we conclude that $\pi \in \mathfrak{r}(M_H)$. Hence all generators of $(\mathfrak{r}(M))_H$ are contained in $\mathfrak{r}(M_H)$ which proves that $(\mathfrak{r}(M))_H = \mathfrak{r}(M_H)$. Let Q be a primary module of F with $\mathfrak{p}$ as radical, then we know from the above that $\mathfrak{p}_H$ is the radical of Q_H. It remains to be shown that, if $\pi \in P[x_0, \cdots, x_n]$, $v \in F$ and $\pi v \in Q_H$, then either $\pi \in \mathfrak{p}_H$ or $v \in Q_H$. First suppose that π is a form and that $\sum_{j=0}^{k} w_j$ is the decomposition of v in vector forms. Then $\sum_{j=0}^{k} \pi w_j$ is the decomposition of πv and, since Q_H is homogeneous, we conclude that $\pi w_j \in Q_H$ for $0 \leq j \leq k$. Since $Q_H \subseteq Q$, either $\pi \in \mathfrak{p}$ and hence $\pi \in \mathfrak{p}_H$ or each $w_j \in Q$ and hence $w_j \in Q_H$ and the lemma is proved for the case that π is a form. If v is a vector form and π is an arbitrary polynomial, the lemma is proved in the same way. Consequently, we make the induction hypothesis that Lemma 2.11 is proved when either the number of homogeneous components of π is less than k or the number of homogeneous components of v is less than σ. Let $\pi = \sum_{j=0}^{k} \phi_j$ be the decomposition of π in forms and $v = \sum^{\sigma} w_j$ the decomposition of v in vector forms, where j is the degree of ϕ_j and of w_j and where $\phi_k \neq 0$, $w_\sigma \neq 0$. Then, $\phi_k w_\sigma$ is a homogeneous component

of πv, namely the one of highest degree, and hence $\phi_k w_\sigma \in Q_H$. If $\phi_k \notin \mathfrak{p}_H$, we conclude from the induction hypothesis that $w_\sigma \in Q_H$ and hence that $\pi \sum_{j=0}^{\sigma-1} w_j \in Q_H$. Since $\phi_k \notin \mathfrak{p}_H$ implies $\pi \notin \mathfrak{p}_H$, it follows from the induction hypothesis that $\sum_{j=0}^{\sigma-1} w_j \in Q_H$ and hence that $v = w_\sigma + \sum_{j=0}^{\sigma-1} w_j$ is contained in Q_H. If $\phi_k \in \mathfrak{p}_H$ and $v \notin Q_H$, we can find an integer ρ such that $\phi_k{}^\rho v \in Q_H$ and $\phi_k{}^{\rho-1} v \notin Q_H$ where ρ may be $1 (\phi_k{}^0 v = v)$. We then conclude that $(\sum_{j=0}^{k-1} \phi_j)(\phi_k{}^{\rho-1} v) \in Q_H$ and hence, from the induction hypothesis, that $\sum_{j=0}^{k-1} \phi_j \in \mathfrak{p}_H$. This makes $\pi = \phi_k + \sum_{j=0}^{k-1} \phi_j$ an element of $\mathfrak{p}_H$ and the lemma is proved.

LEMMA 2.12. *Let M be an H-module of F. Then we can find a Noether decomposition $M = \bigcap_{j=1}^{h} Q_j$ of M where all Q_j for $1 \leq j \leq h$ are H-modules. Consequently, the associated primes of an H-module are homogeneous ideals.*

Proof. Let $M = \bigcap_{j=1}^{h} Q_j$ be any Noether decomposition of M. Since M is homogeneous, $M = M_H$, and, since the homogeneous kernel of an intersection is always the intersection of the kernels of the components, $M = \bigcap_{j=1}^{h} (Q_j)_H$. According to Lemma 2.11, $(Q_j)_H$ is primary and hence from $\bigcap_{j=1}^{h} (Q_j)_H$ we can obtain a Noether decomposition of M by grouping together those $(Q_j)_H$ which have the same radicals and by deleting the superfluous ones. Since every Noether decomposition of M must have exactly h components, it follows that $M = \bigcap_{j=1}^{h} (Q_j)_H$ itself is a Noether decomposition of M. Since $(Q_j)_H$ is a homogeneous module whose radical is hence homogeneous, Lemma 2.12 is proved.

Let $\mathfrak{p}$ be a prime ideal of $P[x_0, \cdots, x_n]$; let $M \subseteq F$; and let the $\mathfrak{p}$-closure $M(\mathfrak{p})$ be defined as in [1] Section **2.3**. Then, it is in general not true that $(M(\mathfrak{p}))_H = M_H(\mathfrak{p}_H)$. For example, let $P[x_0, \cdots, x_n] = P[x_0, x_1]$; let the dimension of F be one, i.e., $F = P[x_0, x_1]$; let M be the ideal $(x_0 - 1)$; and let $\mathfrak{p}$ be the ideal (x_1). Then, $M(\mathfrak{p}) = P[x_0, x_1]$ and hence $(M(\mathfrak{p}))_H = P[x_0, x_1]$. However, $M_H = (0)$; $\mathfrak{p}_H = (x_1)$; and hence $M_H(\mathfrak{p}_H) = (0)$. On the other hand, we assert that it is always true that, if M is $\mathfrak{p}$-closed (see [1] Section **2.3**), M_H is $\mathfrak{p}_H$-closed. To say that M is $\mathfrak{p}$-closed is equivalent to saying that the associated primes of M, say $\mathfrak{p}_1, \cdots, \mathfrak{p}_h$, are all contained in $\mathfrak{p}$. This implies that $(\mathfrak{p}_j)_H \subseteq \mathfrak{p}_H$ for $1 \leq j \leq h$. From the proof of Lemma 2.12, it follows that the associated primes of M_H are among the

$(\mathfrak{p}_j)_H$ for $1 \leq j \leq h$ and hence that they are all contained in $\mathfrak{p}_H$, which proves the assertion. If we take for $\mathfrak{p}$ the zero ideal, $\mathfrak{p}_H$ is also the zero ideal, and hence, if a module $M \subseteq F$ is closed (see [1] Section **2.7**), then M_H is closed. Furthermore, if M is an H-module, Lemma 2.12 and the uniqueness of the isolated components of M (see [1] Section **2.2**) imply that *isolated components of H-modules are always H-modules.* In particular, *all* $\mathfrak{C}$*-closures* (see [1] Section **2.3**) *of a homogeneous module are always homogeneous modules.*

It is important to notice that in the present section we have considered the H-module M as imbedded in the complete vector space F. If $M \subseteq N$, where N is also an H-module of F, *this section undergoes no change whatsoever if* M *is considered as imbedded in* N *instead of in* F. In [1], the Noether decomposition of M as a submodule of any Noetherian vector space, not necessarily F, was considered. *The definitions, lemmas and proofs of this section remain valid when the Noetherian vector space* N *is considered as the imbedding space of* M.

2.2. ***H*-modules of *F* and modules of *V*.** The notation is the same as in the previous section. If $v \in F$, $\bar{v}$ denotes the vector of V which arises from v by carrying out the substitution $(X_0)\, x_0 = 1$ in all the components of v. In the same way, the polynomial $\bar{\pi} \in P[x_1, \cdots, x_n]$ arises from $\pi \in P[x_0, \cdots, x_n]$ by carrying out (X_0). Since $\overline{v_1 + v_2} = \bar{v}_1 + \bar{v}_2$ and $\bar{\pi}\bar{v} = \overline{\pi v}$, (X_0) transforms a module $M \subseteq F$ in a module $\bar{M} \subseteq V$ and an ideal $\mathfrak{c} \subseteq P[x_0, \cdots, x_n]$ in an ideal $\mathfrak{c} \subseteq P[x_1, \cdots, x_n]$ (see [7] Section 3 or [8] Section 4). As for ideals, it follows immediately that, if $v_1, \cdots, v_s$ are generators of M, $\bar{v}_1, \cdots, \bar{v}_s$ are generators of $\bar{M}$ and hence that $\overline{(M_1, M_2)} = (\bar{M}_1, \bar{M}_2)$ and $\overline{\mathfrak{c}M} = \bar{\mathfrak{c}}\bar{M}$. If M is an H-module of F, the substitution (X_0) has to be carried out only in the vector forms of M in order to obtain $\bar{M}$ (the proof is the same as for ideals, given in [7], p. 506). If $L \subseteq V$, L_0 always denotes the H-module of F which is generated by all the vector forms of F which are transformed by (X_0) in vectors of L. L_0 is called the *equivalent H-module* of L and is the largest H-module of F which is transformed in L by (X_0).

LEMMA 2.21. *Let* M *be an* H*-module of* F, $\bar{M}$ *the corresponding module of* V, *and* M_0 *the equivalent* H*-module of* $\bar{M}$. *Then,* M_0 *consists of all the vectors* v *of* F *for which there exists an integer* $\rho \geq 0$ *such that* $x_0^\rho v \in M$. *Hence,* M_0 *is the isolated component of* M *whose associated primes contain no power of* x_0.

Proof. If v_1 and v_2 are vector forms of F where $\partial(v_1) \geq \partial(v_2)$, then $\overline{v}_1 = \overline{v}_2$ if and only if $v_1 = x_0^{\rho} v_2$ where $\rho = \partial(v_1) - \partial(v_2)$. It is clear that, if $v_1 = x_0^{\rho} v_2$, then $\overline{v}_1 = \overline{v}_2$; hence suppose that $\overline{v}_1 = \overline{v}_2$. Since the components of v_1 and v_2 are forms, it follows from [7], p. 507, that there exist non-negative integers $\rho_1, \cdots, \rho_m$ and $\sigma_1, \cdots, \sigma_m$ such that

$$v_1 \begin{pmatrix} x_0^{\rho_1} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & x_0^{\rho_m} \end{pmatrix} = v_2 \begin{pmatrix} x_0^{\sigma_1} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & x_0^{\sigma_m} \end{pmatrix}.$$

This implies that $\sigma_i + \partial(v_2) = \rho_i + \partial(v_1)$ for each $1 \leq i \leq m$ and hence that $\partial(v_1) + \partial(v_2) = \sigma_i - \rho_i = \rho$. Consequently, $v_1 = x_0^{\rho} v_2$. The fact that M_0 consists of all vectors of F for which $x_0^{\rho} v \in M$ for some $\rho \geq 0$ can then be proved as for ideals (see [7] Theorem 8). In the terminology of [1], Section **2.3**, this means that M_0 is the $\mathfrak{C}$-closure of M where $\mathfrak{C}$ is the multiplicatively closed subset of $P[x_0, \cdots, x_n]$ which consists of all powers of x_0. The rest of Lemma 2.21 is a corollary of Lemma 2.31 of [1].

Let (T) denote a non-singular, linear transformation which transforms the variables $x_0, \cdots, x_n$ in the variables $y_0 = T(x_0), \cdots, y_n = T(x_n)$. The transformation (T) establishes an isomorphism T between the m-dimensional row vector space F over $P[x_0, \cdots, x_n]$ and the m-dimensional row vector space $T(F)$ over $P[y_0, \cdots, y_n]$. If $v \in F$ then $T(v)$ is the vector of $T(F)$ which arises from v by carrying out the substitution (T) in the components of v. Since T is an isomorphism, a module $M \subseteq F$ is transformed in a module $T(M) \subseteq T(F)$ and, if $\mathfrak{p}_1, \cdots, \mathfrak{p}_h$ are the associated primes of M, $T(\mathfrak{p}_1), \cdots, T(\mathfrak{p}_h)$ are the associated primes of $T(M)$. If $\mathfrak{p}_1, \cdots, \mathfrak{p}_{h-1}$ are all different from the ideal $(x_0, \cdots, x_n)$ and P is infinite, (T) can be chosen in such a way that no power x_0^{ρ} is contained in any $T(\mathfrak{p}_j)$ for $1 \leq j \leq h-1$ (see [7] p. 507). This proves the following lemma.

LEMMA 2.22. *If P is infinite, after a suitable non-singular linear transformation the M_0 of Lemma 2.21 is the isolated component of the M of Lemma 2.21 whose associated primes are different from $(x_0, \cdots, x_n)$.*

In other words, after a suitable linear transformation, either $M_0 = M$, or M has a primary component whose radical is $(x_0, \cdots, x_n)$, in which case M_0 is obtained from a Noether decomposition of M by omitting that primary component.

In the same way, the remainder of [7], Section 3, and [8], part I, can be extended to polynomial modules. We shall use the following lemma whose proof, being a direct extension of the proof for the corresponding lemma for ideals, is omitted. For any module $M \subseteq V$ and prime ideal $\mathfrak{p} \subseteq P[x_1, \cdots, x_n]$, the modules $M'(\mathfrak{p})$ and $M(\mathfrak{p})$ are defined as in [1], Sections **2. 3** and **2. 5**, and the lower index 0 always denotes the equivalent H-module. In the same way, if $\mathfrak{C}$ is any multiplicatively closed subset of $P[x_1, \cdots, x_n]$ which does not contain the zero element, $M(\mathfrak{C})$ is defined in [1] Section **2. 3** and $\mathfrak{C}_0$ denotes the multiplicatively closed subset of all forms of $P[x_0, \cdots, x_n]$ which are transformed in polynomials of $\mathfrak{C}$ by the substitution (X_0).

LEMMA 2. 23. *Let* $M = \bigcap_{j=1}^{h} Q_j$ *be a Noether decomposition of the module* $M \subseteq V$ *where* $\mathfrak{p}_j$ *is the radical of* Q_j. *Then,* $M_0 = \bigcap_{j=1}^{h} (Q_{j0})$ *is a Noether decomposition of* M_0 *and* $\mathfrak{p}_{j0}$ *is the radical of* Q_{j0}. *It follows that, if* Q *is a primary (or prime) module of* V *with* $\mathfrak{p}$ *as radical,* Q_0 *is a primary (or prime) module of* F *with* $\mathfrak{p}_0$ *as radical. It furthermore follows that always* $(M'(\mathfrak{p}))_0 = M'_0(\mathfrak{p}_0)$, $(M(\mathfrak{p}))_0 = M_0(\mathfrak{p}_0)$ *and* $(M(\mathfrak{C}))_0 = M_0(\mathfrak{C}_0)$.

For any H-module M of F, the *reduced dimension* $d(M)$ is the largest reduced dimension of its *non-zero* associated primes. If M has only the zero ideal of $P[x_0, \cdots, x_n]$ as associated prime, i. e. if M is closed (see [1] Section **2. 7**), $d(M)$ is considered as -1. Since, according to Lemma 2. 12, the associated primes of M are H-ideals of $P[x_0, \cdots, x_n]$, the above definition of $d(M)$ is complete (see [7] p. 510 for the definition of reduced dimension of an H-ideal).

Again, it is important to observe that in the present section we have used as the imbedding Noetherian vector space of an H-module M, the vector space F, and of a non-homogeneous module $\bar{M}$ the vector space V. *If* N *is any homogeneous module of* F *where* $N = N_0$ *and* $M \subseteq N$, *all definitions, lemmas and proofs of this section remain valid when* N *is considered as the imbedding Noetherian factor space of* M *and* $\bar{N}$ *as the imbedding space of* $\bar{M}$.

2. 3. The Hilbert characteristic function of a polynomial module. In this section it is shown how the work of B. L. v. d. Waerden on the Hilbert characteristic function (see [7], Section 4 and [8], part II) can be extended to polynomial modules. The motivation for this study is the fact that it enables us to investigate the exponents $\lambda(\mathfrak{p})$, $\delta(\mathfrak{p})$ and $l(\mathfrak{p})$ (see Remark 1. 23). The notation is the same as in the previous sections. P-linear

dependence and P-linear independence means linear dependence and independence with respect to P. There exists a one to one correspondence between the vector forms of degree ρ of F and the vectors of the $m(n+\rho-1)!/n!(\rho-1)!$-dimensional row vector space Φ over P as scalar domain. Since this correspondence is an operator isomorphism with respect to P as operator domain, the notion of P-linear dependence for vector forms of degree ρ of F follows the rules of P-linear dependence for vectors of Φ. If M is an H-module of F, $\phi(\rho; M)$ denotes the maximum number of vector forms of degree ρ of M which are P-linearly independent. It follows from the above remark that, for two H-modules L and M,

$$\phi(\rho; (L, M)) = \phi(\rho; L) + \phi(\rho; M) - \phi(\rho; L \cap M)$$

(designated equation I). If $M \subseteq N$ are two H-modules of F, $\chi(\rho; N/M)$ denotes the maximum number of vector forms of degree ρ of N which are P-linearly independent mod. M. As above, we conclude that

$$\chi(\rho; N/M) = \phi(\rho; N) - \phi(\rho; M)$$

(designated equation II). From equations I and II, it follows that, if $(L, M) \subseteq N$ where L is also an H-module of F, then

$$\chi(\rho; N/(L, M)) = \chi(\rho; N/L) + \chi(\rho; N/M) - \chi(\rho; N/L \cap M)$$

(designated equation III see [7] p. 511 for the corresponding proof for ideals).

LEMMA 2.31. *Let $\psi \in P[x_0, \cdots, x_n]$ be a form of degree γ and $L \subseteq M$ be two H-modules of F. Let ψ be M-relatively prime to L, i.e., if $\psi v \in L$ where v is a vector form of M then $v \in L$. Then,*

$$\chi(\rho; M/(L, \psi M)) = \chi(\rho; M/L) - \chi(\rho - \gamma; M/L).$$

Proof. The product ψM of a form and a module is defined in [1] Section **2.1.** If v is a vector form of degree ρ of ψM, then $v = \psi m$ where m is a vector form of degree $\rho - \gamma$ of M and hence $\phi(\rho; \psi M) = \phi(\rho - \gamma; M)$. If v is a vector form of degree ρ of $L \cap \psi M$, then $v = \psi m$ where m is a vector form of degree $\rho - \gamma$ of M and where, furthermore, $\psi m \in L$ and hence $m \in L$. It follows that $\phi(\rho; L \cap \psi M) = \phi(\rho - \gamma; L)$. Then, from equation III, it follows that

$$\chi(\rho; M/(L, \psi M)) = \chi(\rho; M/L) + \chi(\rho; M/\psi M) - \chi(\rho; M/L \cap \psi M)$$

and, from equation II, it follows that $\chi(\rho; M/\psi M) = \phi(\rho; M) - \phi(\rho; \psi M)$ and $\chi(\rho; M/L \cap \psi M) = \phi(\rho; M) - \phi(\rho; L \cap \psi M)$. Consequently,

$$\chi(\rho; M/(L, \psi M)) = \chi(\rho; M/L) + \phi(\rho - \gamma; L) - \phi(\rho - \gamma; M)$$

and hence according to equation II,

$$\chi(\rho; M/(L, \psi M)) = \chi(\rho; M/L) - \chi(\rho - \gamma; M/L) .$$

As in [1], Section **2. 3**, let $\mathfrak{C}$ denote a multiplicatively closed subset of $P[x_0, \cdots, x_n]$ which does not contain the zero element ω of $P[x_0, \cdots, x_n]$ and let $M(\mathfrak{C})$ denote the $\mathfrak{C}$-closure of the H-module M of F. Since $M(\mathfrak{C})$ is an isolated component of M or is equal to F, a Noether decomposition of M gives rise to a decomposition $M = M(\mathfrak{C}) \cap Q_1 \cap \cdots \cap Q_h$ where the radical $\mathfrak{p}_j$ of Q_j, $1 \leq j \leq h$, is an H-ideal of $P[x_0, \cdots, x_n]$ which contains an element of $\mathfrak{C}$. According to [1], Lemma 2. 21, if $Q^*_j = M(\mathfrak{C}) \cap Q_j$ for $1 \leq j \leq h$, $M = Q^*_1 \cap \cdots \cap Q^*_h$ is a Noether decomposition of M as a submodule of the Noetherian vector space $M(\mathfrak{C})$ and the radical of Q^*_j as a primary submodule of $M(\mathfrak{C})$ is $\mathfrak{p}_j$.

LEMMA 2. 32. *Let M be an H-module of F and let the reduced dimension d of M as a submodule of the Noetherian vector space $M(\mathfrak{C})$ be -1. Then, for large enough ρ, $\chi(\rho; M(\mathfrak{C})/M) = 0$.*

Proof. Let, as above, $M = M(\mathfrak{C}) \cap Q_1 \cap \cdots \cap Q_h$. Since the associated primes of M, considered as a submodule of $M(\mathfrak{C})$, are $\mathfrak{p}_1, \cdots, \mathfrak{p}_h$, h is at most 2 and $\mathfrak{p}_1 = (x_0, \cdots, x_n)$ and $\mathfrak{p}_2 = (\omega)$. However, if $\mathfrak{p}_2 = (\omega)$, $M(\mathfrak{C})$ could not be isolated or equal to F (see [1] Sections **2. 3** and **2. 7**) and hence $h = 1$ and $p_1 = (x_0, \cdots, x_n)$. It follows that, for large enough ρ,

$$\mathfrak{p}_1{}^{\rho} F \cap M(\mathfrak{C}) \subseteq Q_1 \cap M(\mathfrak{C}) = M$$

which means that for large enough ρ the vector forms of degree ρ of $M(\mathfrak{C})$ are contained in M and hence that $\chi(\rho; M(\mathfrak{C})/M) = 0$.

THEOREM 2. 31. *Let M be an H-module of F whose reduced dimension as a submodule of the Noetherian vector space $M(\mathfrak{C})$ is d. Then, for large enough ρ, $\chi(\rho; M(\mathfrak{C})/M) = a_0 \binom{\rho}{d} + a_1 \binom{\rho}{d-1} + \cdots + a_d$, where $\binom{\rho}{d-j} = \rho!/(d-j)!(\rho - d + j)!$ and where $a_0, \cdots, a_d$ are whole numbers independent of ρ.*

Proof. As in the proof of [7], Theorem 17, we observe that, when $d = -1$, Theorem 2. 31 follows from Lemma 2. 32. Hence, we make the induction hypothesis that Theorem 2. 31 has been proved for $d = -1, 0, \cdots, d-1$ and we assume that the H-module M has reduced dimension d as a submodule of $M(\mathfrak{C})$. We first consider the case when M is primary as a

submodule of $M(\mathfrak{E})$, i. e., when $M = M(\mathfrak{E}) \cap Q$ where $\mathfrak{p}$ is the associated prime of Q. Then $\mathfrak{p}$ cannot be (ω) since $M(\mathfrak{E})$ could then be neither isolated nor equal to F. Furthermore, $(x_0, \cdots, x_n) \not\subseteq \mathfrak{p}$ for, since $(x_0, \cdots, x_n)$ is a maximal prime ideal, this would imply that $\mathfrak{p} = (x_0, \cdots, x_n)$ and hence that $d = -1$. Consequently, there exists a form $\psi \in P[x_0, \cdots, x_n]$ of degree one where $\psi \in \mathfrak{p}$. Since $\psi v \in Q$ implies $v \in Q$ we conclude, from Lemma 2.31, that

$$\chi(\rho; M(\mathfrak{E})/(M, \psi M(\mathfrak{E}))) = \chi(\rho; M(\mathfrak{E})/M) - \chi(\rho - 1; M(\mathfrak{E})/M)$$

(designated equation IV). We assert that, if $L = (M, \psi M(\mathfrak{E}))$, then $L(\mathfrak{E}) = M(\mathfrak{E})$ and the reduced dimension of L as a submodule of the Noetherian vector space $M(\mathfrak{E})$ is less than d. Since $M \subseteq L$, $M(\mathfrak{E}) \subseteq L(\mathfrak{E})$; and, since $L \subseteq M(\mathfrak{E})$ and $M(\mathfrak{E})$ is $\mathfrak{E}$-closed (see [1] Lemma 2.31), $L(\mathfrak{E}) \subseteq M(\mathfrak{E})$ which proves the first part of the assertion. Consequently we can again write $L = M(\mathfrak{E}) \cap Q'_1 \cap \cdots \cap Q'_k$ where Q'_j is a primary module of F which has $\mathfrak{p}'_j$ as radical, and where, as before, $\mathfrak{p}'_1, \cdots, \mathfrak{p}'_k$ are the associated primes of L as a submodule of $M(\mathfrak{E})$. Since $\psi M(\mathfrak{E}) \subseteq L \subseteq Q'_j$ and $M(\mathfrak{E}) \not\subseteq Q'_j$, $\psi \in \mathfrak{p}'_j$ for $1 \leq j \leq k$. Since $M = M(\mathfrak{E}) \cap Q$ and since, for large enough ρ, $\mathfrak{p}^\rho M(\mathfrak{E}) \subseteq M(\mathfrak{E}) \cap Q$, we conclude that, for large enough ρ, $\mathfrak{p}^\rho M(\mathfrak{E}) \subseteq L \subseteq Q'_j$ and hence $\mathfrak{p} \subseteq \mathfrak{p}'_j$ for $1 \leq j \leq h$. Hence, $\mathfrak{p} \subset (\mathfrak{p}, \psi) \subseteq \mathfrak{p}'_j$ for $1 \leq j \leq h$ which shows that the reduced dimension of each $\mathfrak{p}'_j$ is less than d and the assertion is proved. We then conclude from the induction hypothesis that for large enough ρ

$$\chi(\rho; M(\mathfrak{E})/(M, \psi M(\mathfrak{E}))) = a_0\binom{\rho}{d-1} + a_1\binom{\rho}{d-2} + \cdots + a_{d-2}\binom{\rho}{1} + a_{d-1}$$

where $a_0, \cdots, a_{d-1}$ are whole numbers. If this expression is substituted in equation IV, we obtain a recursion formula for $\chi(\rho; M(\mathfrak{E})/M)$ from which, exactly as in [7], p. 513. it follows that

$$\chi(\rho; M(\mathfrak{E})/M) = a_0\binom{\rho}{d} + A_1\binom{\rho}{d-1} + \cdots + A_{d-1}\binom{\rho}{1} + A_d$$

where $A_1, \cdots, A_d$ are whole numbers and where a_0 has the same meaning as above and hence is non-negative. Hence, Theorem 2.31 is proved for the special case when $h = 1$ in the decomposition $M = M(\mathfrak{E}) \cap Q_1 \cap \cdots \cap Q_h$. We now make the induction hypothesis that Theorem 2.31 has been proved, not only for $d = -1, 0, \cdots, d-1$, but also for $h = 1, 2, \cdots, h-1$. We then observe that $M = L \cap N$ where

$$L = M(\mathfrak{E}) \cap Q_1 \cap \cdots \cap Q_{k-1} \cap Q_{k+1} \cap \cdots \cap Q_h \text{ and } N = M(\mathfrak{E}) \cap Q_k$$

and where Q_k is any one of the primary components $Q_1, \cdots, Q_h$ whose

reduced dimension is exactly d. From equation III we conclude that (1) $\chi(\rho; M(\mathfrak{C})/M) = \chi(\rho; M(\mathfrak{C})/L) + \chi(\rho; M(\mathfrak{C})/N) - \chi(\rho; M(\mathfrak{C})/(L, N))$. We then prove as before that $L(\mathfrak{C}) = N(\mathfrak{C}) = (L, N)(\mathfrak{C}) = M(\mathfrak{C})$ and that the reduced dimension of (L, N) as a submodule of the Noetherian vector space $M(\mathfrak{C})$ is less than d. Consequently, according to the induction hypothesis, Theorem 2.31 is proved for each term of the right hand side of (1) and hence the theorem is proved for $\chi(\rho; M(\mathfrak{C})/M)$.

The polynomial $\chi(\rho; M(\mathfrak{C})/M) = \sum_{j=0}^{d} a_j \binom{\rho}{d-j}$ will be called the *characteristic function of M with respect to $M(\mathfrak{C})$*. The coefficient a_0 will be called the *degree of M with respect to $M(\mathfrak{C})$*. We shall talk about the $M(\mathfrak{C})$-characteristic function, the $M(\mathfrak{C})$-degree and the $M(\mathfrak{C})$-reduced dimension of M.

THEOREM 2.32. *If, as before, $M = M(\mathfrak{C}) \cap Q_1 \cap \cdots \cap Q_h$ and d is the $M(\mathfrak{C})$-reduced dimension of the H-module M, then the $M(\mathfrak{C})$-degree of M is the sum of the $M(\mathfrak{C})$-degrees of those components $M(\mathfrak{C}) \cap Q_k$ where the radical $\mathfrak{p}_k$ of Q_k has reduced dimension d.*

Proof. We use the notation of the proof of Theorem 2.31. Since the $M(\mathfrak{C})$-reduced dimension of (L, N) is less than d, the $M(\mathfrak{C})$-degree of M is equal to the sum of the $M(\mathfrak{C})$-degrees of L and N, according to equation (1). Since N is already of the form $M(\mathfrak{C}) \cap Q_k$, where Q_k has reduced dimension d, we have to prove Theorem 2.32 only for L. Since L, considered as a submodule of $M(\mathfrak{C})$, has only $h-1$ components, Theorem 2.32 is proved by induction as soon as the theorem is proved for $h=1$. However, for $h=1$ Theorem 2.32 is trivial.

THEOREM 2.33. *If the $M(\mathfrak{C})$-reduced dimension d of the H-module M satisfies $d > -1$, the $M(\mathfrak{C})$-degree a_0 of M satisfies $a_0 > 0$.*

Proof. If $d = 0$, according to Theorem 2.31 $\chi(\rho; M(\mathfrak{C})/M) = a_0$. Then, $a_0 = 0$ would imply that for large enough ρ all the vector forms of degree ρ of $M(\mathfrak{C})$ are contained in M, i.e., that $\mathfrak{p}^\rho F \cap M(\mathfrak{C}) \subseteq M$ where $\mathfrak{p} = (x_0, \cdots, x_n)$. We would conclude that $\mathfrak{p}^\rho M(\mathfrak{C}) \subseteq M$ and hence, as in the proof of Theorem 2.31, that $\mathfrak{p}$ is contained in every associated prime of M as a submodule of $M(\mathfrak{C})$. It would follow that $M = M(\mathfrak{C}) \cap Q_1$ where $\mathfrak{p}$ is the radical of the primary module Q_1 and hence that $d = -1$. Hence, when $d = 0$, $a_0 \neq 0$ and, since a_0 represents a number of vectors, $a_0 > 0$. Hence, we assume that Theorem 2.33 has been proved for $M(\mathfrak{C})$-reduced dimensions $0, 1, \cdots, d-1$ and that the $M(\mathfrak{C})$-reduced dimension of M is d.

It follows from Theorem 2.32 that Theorem 2.33 has to be proved only for the case $h = 1$. According to the proof of Theorem 2.31, if $h = 1$, the $M(\mathfrak{C})$-degree of M is equal to the coefficient of $\binom{\rho}{d-1}$ of the characteristic function $\chi(\rho; M(\mathfrak{C})/(M, \psi M(\mathfrak{C}))$. Hence, according to the induction hypothesis, Theorem 2.33 is proved as soon as we have proved that the $M(\mathfrak{C})$-reduced dimension of $(M, \psi M(\mathfrak{C}))$ is exactly $d - 1$. The proof of this fact is an immediate extension of [7], p. 514, to modules.

Let M be any module of the m-dimensional row vector space V over $P[x_1, \cdots, x_n]$ as scalar domain. Let $\mathfrak{C}$ be a multiplicatively closed subset of $P[x_1, \cdots, x_n]$ which does not contain the zero element and let M have dimension d as a sub-module of the Noetherian vector space $M(\mathfrak{C})$. This means that, in the representation $M = M(\mathfrak{C}) \cap Q_1 \cap \cdots \cap Q_h$ where, as before, Q_j is a primary module of V with $\mathfrak{p}_j$ as radical, the highest dimension of the prime ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_h$ is d. We know that then $M_0 = M_0(\mathfrak{C}_0) \cap Q_{10} \cap \cdots \cap Q_{h0}$ where the lower index zero denotes the equivalent H-modules. Since the substitution (X_0) establishes a one to one correspondence between the vector forms of degree ρ of M_0 and the vectors of degree $\leq \rho$ of M, the following theorem is a corollary of Theorem 2.31, 2.32 and 2.33.

THEOREM 2.34. *Let the arbitrary module M of V have the dimension d as a proper submodule of the Noetherian vector space $M(\mathfrak{C})$. Then, for ρ sufficiently large, the maximum number of vectors of degree $\leq \rho$ of $M(\mathfrak{C})$ which are I-linearly independent* mod. *M is given by a polynomial $a_0\binom{\rho}{d} + a_1\binom{\rho}{d-1} + \cdots + a_d$ where $a_0, \cdots, a_d$ are whole numbers independent of ρ and where $a_0 > 0$. The $M(\mathfrak{C})$-degree a_0 of M is the sum of the $M(\mathfrak{C})$-degrees of the components $M(\mathfrak{C}) \cap Q_j$, where the Q_j's are those primary components of the representation $M = M(\mathfrak{C}) \cap Q_1 \cap \cdots \cap Q_h$ whose radicals are d-dimensional.*

If M is zero dimensional as a submodule of the vector space $M(\mathfrak{C})$, the polynomial of Theorem 2.34 reduces to a_0. This gives the following theorem.

THEOREM 2.35. *If the arbitrary module $M \subseteq V$ is zero-dimensional as a submodule of the Noetherian vector space $M(\mathfrak{C})$, the factor module $M(\mathfrak{C})/M$ has finite P-rank a_0. That is, $M(\mathfrak{C})/M$ is a Noetherian vector space over P as scalar domain which is generated by a_0 elements. This rank a_0 is the sum of the finite P-ranks of the factor modules $M(\mathfrak{C})/M(\mathfrak{C}) \cap Q_j$ where the $M(\mathfrak{C}) \cap Q_j$ are the primary components of M is a submodule of $M(\mathfrak{C})$.*

If M is a module of V, the $M(\mathfrak{C})$-characteristic function and the $M(\mathfrak{C})$-degree of M denote the $M_0(\mathfrak{C}_0)$-characteristic function and $M_0(\mathfrak{C}_0)$-degree

of M_0. Hence, the $M(\mathfrak{C})$-characteristic function of a module $M \subseteq V$ is, for ρ sufficiently large, the maximum number of vectors of degree $\leq \rho$ of $M(\mathfrak{C})$ which are P-linearly independent mod. M. The same terminology will be used for ideals of $P[x_1, \cdots, x_n]$.

2.4. Connection between length and degree. Let M be any module of the vector space V and let M be $\mathfrak{p}$-primary as a submodule of the Noetherian vector space $M(\mathfrak{C})$. (As in [1], $\mathfrak{p}$-primary means primary with $\mathfrak{p}$ as radical.) We know from [1], Theorem 2.41, that there exists a $\mathfrak{p}$-primary composition sequence of length l, $M \subset M_1 \subset \cdots \subset M_{l-1} \subset M(\mathfrak{C})$. If $\mathfrak{p}$ is zero-dimensional, $\mathfrak{p}$ is maximal and hence, according to [1], Theorem 2.41, the factor module M_{i+1}/M_i is then operator isomorphic with $P[x_1, \cdots, x_n]/\mathfrak{p}$ with respect to $P[x_1, \cdots, x_n]$ as operator domain and hence certainly with respect to P as operator domain. The factor module $P[x_1, \cdots, x_n]/\mathfrak{p}$ is an extension field of P of finite rank a^*_0, where this rank is the degree, in the sense of the Hilbert characteristic function, of $\mathfrak{p}$ as a sub-ideal of $P[x_1, \cdots, x_n]$. It follows immediately that *the degree a_0 of M as a $\mathfrak{p}$-primary submodule of $M(\mathfrak{C})$ where $\mathfrak{p}$ is 0-dimensional satisfies $a_0 = la^*_0$.* It is the purpose of this section to extend this statement to the case when $\mathfrak{p}$ is d-dimensional (for ideals, see [8] Sections 29 through 32).

Let t be a new variable and let $V(t)$ denote the m-dimensional row vector space over $P(t)[x_1, \cdots, x_n]$ as scalar domain. If M is a module of V, $M(t)$ denotes the module $V(t)$ which is generated by the vectors of M.

LEMMA 2.41. *Let M be a $\mathfrak{p}$-primary module as a submodule of the Noetherian vector space $M(\mathfrak{C})$. Let $\xi_1, \cdots, \xi_n$ be a general point of $\mathfrak{p}$ where ξ_1 is transcendent with respect to P. Then, $M^* = (M(t), (x_1 - t)M(t)(\mathfrak{C}))$ is $(\mathfrak{p}(t), x_1 - t)$-primary as a submodule of $M^*(\mathfrak{C})$. Furthermore, the length of M as a submodule of $M(\mathfrak{C})$ is equal to the length of M^* as a submodule of $M^*(\mathfrak{C})$.*

Proof. As in [8], Section 30, we assert that $V \cap M^* = M$. All we have to show is that $V \cap M^* \subseteq M$. However, if $v \in V \cap M^*$, then $\phi(t)v = \Sigma_j \pi_j(t) m_j + (x_1 - t)w$ where $\phi(t)$, $\pi_j(t)$ and the components of the vector $w \in V(t)$ are polynomials of $P[t]$ and where $m_j \in M$. Hence, for $t = x_1$ we conclude that $\phi(x_1)v = \Sigma_j \pi_j(x_1) m_j \in M$. Since ξ_1 is transcendental with respect to P, $\phi(x_1) \notin \mathfrak{p}$ and hence $v \in M$ which proves the assertion. In the same way, the remainder of the proof of Lemma 2.41 is an immediate extension of [8], Section 30, to modules.

For H-modules of F, Lemma 2.41 can be formulated as follows. Let

$F(t)$ denote the m-dimensional row vector space over $P(t)[x_0, \cdots, x_n]$ as scalar domain. If M is an H-module of F, $M(t)$ denotes the H-module of $F(t)$ which is generated by the vectors of M. Since $M(t)$ is then generated by the vector forms of M, $M(t)$ is certainly homogeneous.

LEMMA 2.42. *Let M be an H-module of F which is $\mathfrak{p}$-primary as a submodule of $M(\mathfrak{C})$. Let $\lambda, \lambda\xi_1, \cdots, \lambda\xi_n$ be a general point of the H-ideal $\mathfrak{p}$ where ξ_1 is transcendent with respect to P. Let $M^* = (M(t), (x_1 - tx_0)M(t)(\mathfrak{C}))$ and $\mathfrak{p}^* = (\mathfrak{p}(t), x_1 - tx_0)$. Then, M^*_0 is $\mathfrak{p}^*_0$-primary as a submodule of $M^*_0(\mathfrak{C})$. Furthermore, the length of M as a submodule of $M(\mathfrak{C})$ is the same as the length of M^*_0 as a submodule of $M^*_0(\mathfrak{C})$.*

Proof. The proof is an immediate extension of [8], Section 31, to modules.

We need one more lemma to obtain the above stated purpose of this section. Let K be an extension field of P. Then V_K denotes the m-dimensional row vector space over $K[x_1, \cdots, x_n]$ as scalar domain and F_K denotes the m-dimensional row vector space over $K[x_0, \cdots, x_n]$ as scalar domain. If $M \subseteq V$, M_K denotes the module of V_K which is generated by the vectors of M and the same notation is used for H-modules of F.

LEMMA 2.43. *Let M be an H-module of F. Then, the $M(\mathfrak{C})$-characteristic function of M is equal to the $M_K(\mathfrak{C})$-characteristic function of M_K. The same holds if M is a module of V.*

Proof. The proof is an immediate extension of the proof of Theorem 23 in [7].

THEOREM 2.41. *Let M be an H-module of F which is $\mathfrak{p}$-primary as a submodule of $M(\mathfrak{C})$. Let the $M(\mathfrak{C})$-degree of M be a_0, the $M(\mathfrak{C})$-length of M be l, and the degree of $\mathfrak{p}$ as an H-ideal of $P[x_0, \cdots, x_n]$, a^*_0. Then, $a_0 = la^*_0$. The same holds if M is a module of V.*

Proof. The proof is an immediate extension of [8], Section 32, to modules.

2.5. Systems of linear equations. In [1], Definition 2.51, the $\mathfrak{p}$-elementary divisor $\mathfrak{e}(\mathfrak{p})$, the $\mathfrak{p}$-exponent $\rho(\mathfrak{p})$ and the $\mathfrak{p}$-length $l(\mathfrak{p})$ of a module $M \subseteq V$ were defined in terms of the modules $M(\mathfrak{p})$ and $M'(\mathfrak{p})$. We now define the $\mathfrak{p}$-degree $a_0(\mathfrak{p})$ in the same way.

DEFINITION 2.51. *Let M be a module of V; let $\mathfrak{p}$ be a prime ideal of*

$P[x_1, \cdots, x_n]$; *and let* $M(\mathfrak{p})$ *and* $M'(\mathfrak{p})$ *be defined as in* [1]. *Then, the* $\mathfrak{p}$*-degree* $a_0(\mathfrak{p})$ *of* M *is the degree of* $M(\mathfrak{p})$ *as a submodule of* $M'(\mathfrak{p})$.

Since $M(\mathfrak{p})$ is $\mathfrak{p}$-primary as a submodule of $M'(\mathfrak{p})$ (see [1] Lemma 2.51), $a_0(\mathfrak{p}) = l(\mathfrak{p})a^*_0$ where a^*_0 is the degree of $\mathfrak{p}$ as an ideal of $P[x_1, \cdots, x_n]$. Hence, for polynomial modules, the criterian of lengths (see [1], Section **2.6**) can be stated as follows.

THEOREM 2.52. *Let* $M_1 \subseteq M_2$ *be two modules of* V. *Then,* $M_1 = M_2$ *if and only if* M_1 *and* M_2 *have the same associated primes and, for each associated prime* $\mathfrak{p}$, *the same* $\mathfrak{p}$*-degree.*

The zero module of the vector space V is a prime module and hence the developments of [1], Section **2.7**, can be used. Since the radical of the zero module is the zero ideal of $P[x_1, \cdots, x_n]$, Theorem 2.71 of [1] becomes the following.

THEOREM 2.53. *Let* $M_1 \subseteq M_2$ *be two modules of* V. *Then,* $M_1 = M_2$ *if and only if* M_1 *and* M_2 *have the same rank, the same non-zero associated primes, and, for each such associated prime* $\mathfrak{p} \neq 0$, *the same* $\mathfrak{p}$*-degree.*

Let $A = (\alpha_{ij})$ be an $m \times s$ matrix, where $\alpha_{ij} \in P[x_1, \cdots, x_n]$ and $i = 1, \cdots, m$ and $j = 1, \cdots, s$. The columns of A generate a module C, called the column space of A, of the m-dimensional column vector space over $P[x_1, \cdots, x_n]$ as scalar domain. The associated primes, $\mathfrak{p}$-length, $\mathfrak{p}$-elementary divisor, $\mathfrak{p}$-exponent, $\mathfrak{p}$-degree and the rank of A are defined as those of C. The criterion of solvability and Theorem 2.72 of [1] then become the following.

THEOREM 2.54. *Let* $\sum_{j=1}^{s} \alpha_{ij} z_j = \gamma_i$, *where* $i = 1, \cdots, m$ *and* α_{ij} *and* γ_i *are polynomials of* $P[x_1, \cdots, x_n]$, *be a system of* m *linear equations for the* s *unknowns* z_j. *Then this system has a solution* $z_j \in P[x_1, \cdots, x_n]$ $(j = 1, \cdots, s)$ *if and only if the* $m \times s$ *matrix* $A = (\alpha_{ij})$ *and the augmented* $m \times (s+1)$ *matrix* $B = (\alpha_{ij}, \gamma_i)$ *have the same associated primes and, for each associated prime* $\mathfrak{p}$, *the same* $\mathfrak{p}$*-degree. This is equivalent to saying that* A *and* B *must have the same rank, the same non-zero associated primes and, for each non-zero associated prime* $\mathfrak{p}$, *the same* $\mathfrak{p}$*-degree.*

The notion of the transformed module M^0 of a module $M \subseteq V$ is discussed in [2]. The resultant of M^0 is investigated in Sections 5 and 6 of [2]. We shall use the term "norm" instead of "resultant" since this polynomial will be shown in Lemma 2.51 to be the complete analogue of the norm for

the case $n = 1$ (see [1], Definition 1.13). If we call the norm of M^0 also the norm of M, Theorem 2.53 of [2] states that the system of linear equations can be solved if and only if A and B have the same rank and norm. The following lemma shows that this criterion of [2] and the present Theorem 2.54 are equivalent.

LEMMA 2.51. *The non-zero associated primes $\mathfrak{p}_1, \cdots, \mathfrak{p}_h$ of a module $M \subseteq V$ are uniquely determined by the irreducible factors $\pi_1, \cdots, \pi_h$ of the norm $\Delta = \Pi^h_{j=1} = \pi_j^{\tau_j}$ of M. The polynomial π_j is the elementary divisor, in the sense of* [2] *Section 5, of $\mathfrak{p}_j$. The degree of the primary factor $\pi_j^{\tau_j}$ of Δ is the $\mathfrak{p}$-degree $a_0(\mathfrak{p})$ of M. If the characteristic p of P is zero, the degree of π_j is the degree a^*_0 of $\mathfrak{p}_j$ and the exponent l_j is then the $\mathfrak{p}_j$-length $l(\mathfrak{p}_j)$ of M.*

Proof. Let $\Pi^h_{j=1}\pi_j^{\tau_j}$ be the factorization of the norm Δ of M. For the case of ideals, that is when the dimension of V is one, it was proved in [9], Sections 3 and 5, that there exists a one to one correspondence between the non-zero associated primes $\mathfrak{p}_1, \cdots, \mathfrak{p}_h$ of the ideal and the irreducible factors $\pi_1, \cdots, \pi_h$ of Δ, where π_j is the elementary divisor of $\mathfrak{p}_j$. These proofs can be extended to modules without difficulty. From [9], Theorem XIII, it then follows that, if $p = 0$, the degree of π_j is the degree of the transform of $\mathfrak{p}_j$ and hence is equal to the degree a^*_0 of $\mathfrak{p}_j$. We know that, for any characteristic $p = 0$ or $p \neq 0$, the $\mathfrak{p}_j$-degree $a_0(\mathfrak{p}_j)$ of M satisfies $a_0(\mathfrak{p}_j) = a^*_0 l(\mathfrak{p}_j)$ where $l(\mathfrak{p}_j)$ is the $\mathfrak{p}_j$-length of M. Hence, all that remains to be proved is that, for arbitrary characteristic of P, the degree of $\pi_j^{\tau_j}$ is $a_0(\mathfrak{p}_j)$. For this purpose, we consider the $n+1$ closures $\mathrm{Cl}_{i-1}(M^0)$, $i = 1, \cdots, n+1$, of the transformed module M^0 which are defined in [2], Section 3. If M is an arbitrary module of V, the intersection of the primary components of dimensions $n-1, n-2, \cdots, n-i+1$ of a Noether decomposition of M is an isolated component of M which is denoted by $\mathrm{Cl}_{i-1}(M)$. It can be easily shown that then $(\mathrm{Cl}_{i-1}(M))^0 = \mathrm{Cl}_{i-1}(M^0)$ where M^0 is the transform of M and $(\mathrm{Cl}_{i-1}(M))^0$ the transform of $\mathrm{Cl}_{i-1}(M)$ (see [9], Section 5, for the proof of the corresponding theorem on ideals which can be immediately extended to modules). *We assert that, if Q is a $\mathfrak{p}$-primary component of M where $\mathfrak{p}$ has dimension $n-i$, then the $\mathfrak{p}$-degree $a_0(\mathfrak{p})$ of M is equal to the degree a'_0 of $Q' = \mathrm{Cl}_{i-1}(M) \cap Q$ as a submodule of $\mathrm{Cl}_{i-1}(M)$* (we omit the lower index j in $\mathfrak{p}_j$). Since Q' is $\mathfrak{p}$-primary as a submodule of $\mathrm{Cl}_{i-1}(M)$, we know that $a'_0 = l' a^*_0$ where l' is the length of Q' as a submodule of $\mathrm{Cl}_{i-1}(M)$ and a^*_0 is the degree of $\mathfrak{p}$. Hence, all we have to show is that $l' = l(\mathfrak{p})$ where $l(\mathfrak{p})$ is the length of $M(\mathfrak{p})$ as a submodule of $M'(\mathfrak{p})$. However,

$Q' = M(\mathfrak{p}) \cap Q_1 \cap \cdots \cap Q_s$ where $Q_1, \cdots, Q_s$ are those primary components of a Noether decomposition of M whose radicals $\mathfrak{p}_1, \cdots, \mathfrak{p}_s$ have dimensions larger than $n-i$ and where $\mathfrak{p}_z \not\subset \mathfrak{p}$ for $1 \leq z \leq s$. In the same way, $\mathrm{Cl}_{i-1}(M) = M'(\mathfrak{p}) \cap Q_1 \cap \cdots \cap Q_s$. Now, let $M(\mathfrak{p}) \subset M_1 \subset \cdots \subset M_{l-1} \subset M'(\mathfrak{p})$ be a $\mathfrak{p}$-primary composition sequence from $M(\mathfrak{p})$ to $M'(\mathfrak{p})$ of length $l = l(\mathfrak{p})$. Then, if $L = Q_1 \cap \cdots \cap Q_s$, $M(\mathfrak{p}) \cap L \subset M_1 \cap L \subset \cdots \subset M_{l-1} \cap L \subset M'(\mathfrak{p}) \cap L$ can be proved immediately to be a composition sequence from Q' to $\mathrm{Cl}_{i-1}(M)$ which proves the assertion. According to [2], M^0 arises from M by adjunction of new variables and an isomorphism from which we conclude that the degree of $\mathrm{Cl}_{i-1}(M^0) \cap Q^0 = Q'^0$ as a submodule of $\mathrm{Cl}_{i-1}(M^0)$ is also $a_0(\mathfrak{p})$ (the upper index 0 denotes as before the transformed module). Let $y_j = \sum_{h=1}^{n} u_{jh} x_h$ be the transformed variables where $1 \leq j \leq n$ and where the n^2 u_{jh} are new variables which are adjoined to P (see [2], Section 2). The module Q'^0 is $\mathfrak{p}^0$-primary as a submodule of $\mathrm{Cl}_{i-1}(M^0) \cap Q^0$ and, if $\eta_1, \cdots, \eta_n$ is a general point of $\mathfrak{p}^0$, then $\eta_{i+1}, \cdots, \eta_n$ are algebraically independent with respect to $P(u)$ and each $\eta_1, \cdots, \eta_i$ is algebraic with respect to $P_\eta = P(u, \eta_{i+1}, \cdots, \eta_n)$ (see [9], Section 3). Every module M^0 gives rise to a module M_η of the m-dimensional row vector space T_η over $P_\eta[y_1, \cdots, y_i]$ as scalar domain. Here, M_η is generated by the vectors of M^0 after the substitution $y_h = \eta_h$ for $i+1 \leq h \leq n$ has been carried out. Then, $(\mathrm{Cl}_{i-1}(M^0))_\eta \cap Q^0{}_\eta = Q'^0{}_\eta$ and $Q'^0{}_\eta$ is $\mathfrak{p}^0{}_\eta$-primary as a submodule of $(\mathrm{Cl}_{i-1}(M^0))_\eta$ where $\mathfrak{p}^0{}_\eta$ is the zero-dimensional prime ideal of $P_\eta[y_1, \cdots, y_i]$ whose general point is $\eta_1, \cdots, \eta_i$ (see [6], Section 97, where the proofs are carried out for ideals). If $Q'^0 \subset H^0{}_1 \subset \cdots \subset H^0{}_{l-1} \subset \mathrm{Cl}_{i-1}(M^0)$ is a $\mathfrak{p}^0$-primary composition sequence from Q'^0 to $\mathrm{Cl}_{i-1}(M^0)$, $Q'^0{}_\eta \subset (H^0{}_1)_\eta \subset \cdots \subset (H^0{}_{l-1})_\eta \subset (\mathrm{Cl}_{i-1}(M^0))_\eta$ can easily be shown to be a $\mathfrak{p}^0{}_\eta$-primary composition sequence from $Q'^0{}_\eta$ to $(\mathrm{Cl}_{i-1}(M^0))_\eta$ and hence the length of $Q'^0{}_\eta$ as a submodule of $(\mathrm{Cl}_{i-1}(M^0))_\eta$ is still $l(\mathfrak{p})$. Since $\mathfrak{p}^0{}_\eta$ is zero-dimensional, the factor module $P_\eta[y_1, \cdots, y_i]/\mathfrak{p}^0{}_\eta$ has finite P_η-rank; *and since* $\mathfrak{p}^0$ *is transformed,* this rank is equal to the degree of $\mathfrak{p}^0$ (this statement is false for not-transformed ideals). Consequently, the degree of $Q'^0{}_\eta$ as a submodule of $(\mathrm{Cl}_{i-1}(M^0))_\eta$ is still $a_0(\mathfrak{p})$. Since $Q'^0{}_\eta$ is zero dimensional as a submodule of $(\mathrm{Cl}_{i-1}(M^0))_\eta$, we conclude that $a_0(\mathfrak{p})$ is the finite P_η-rank of the factor module $(\mathrm{Cl}_{i-1}(M^0))_\eta / Q'^0{}_\eta$. In [9], Sections 3 and 5, it was proved for the case of ideals that the degree of the primary factor $\pi_j^{\tau_j}$ of Δ is the P_η-rank of $(\mathrm{Cl}_{i-1}(M^0))_\eta / Q'^0{}_\eta$, which proof carries over to modules. Hence, Lemma 2.51 is proved.

Remark 2.51. According to Lemma 2.51, $\tau_j = l(\mathfrak{p}_j)$ only if $p = 0$. However we know that, if $n = 1$, always $\tau_j = l(\mathfrak{p}_j)$ for any characteristic

(see [1], Definition 1.13). The reason for this is that the dimension $n-i$ of $\mathfrak{p}_j$ is zero if $n=1$ and hence $i=1$. Theorem XIV of [9] implies that, if $i=1$, Theorem XIII of [9] still holds and hence $\tau_j = l(\mathfrak{p}_j)$. Consequently, we can say that, if $\mathfrak{p}_j$ is an $(n-1)$-dimensional associated prime of M, $\tau_j = l(\mathfrak{p}_j)$ for any characteristic.

Let us call the norm of the column space of a matrix the norm of the matrix. Then, to say that A and B of Theorem 2.54 have the same rank and norm is the same as saying that A and B have the same rank, non-zero associated primes and $\mathfrak{p}$-degrees, which shows the equivalence of Theorem 5.3 of [2] and the present Theorem 2.54. Therefore, Theorem 5.3 of [2] is a consequence of the "criterion of lengths" of [1].

2.6. The elementary divisor, the $\mathfrak{p}$-exponent and the exponents of partial differentiation. The notation is the same as in the proof of Lemma 2.51. The $\mathfrak{p}$-elementary divisor $\mathfrak{e}(\mathfrak{p})$ of M is defined in [1], Section 2.5. The elementary divisor of M (without $\mathfrak{p}$) denotes the elementary divisor of M^0 in the sense of Section 5 of [2]. The elementary divisor E of M has the same irreducible factors as the resultant, i.e., $E = \Pi^h_{j=1}\pi_j^{\sigma_j}$ (see [2], Section 5).

LEMMA 2.61. *The primary factor $\pi_i^{\sigma_i}$ of E which corresponds to the associated prime $\mathfrak{p}$ of M is the elementary divisor of the $\mathfrak{p}$-elementary divisor $\mathfrak{e}(\mathfrak{p})$ of M. If the characteristic of P is zero, σ_i is the $\mathfrak{p}$-exponent $\rho(\mathfrak{p})$ of M.*

Proof. The primary factor $\pi_i^{\sigma_i}$ is the highest common factor of the polynomials contained in $P(u)[y_i, \cdots, y_n] \cap (Q'^0 : \mathrm{Cl}_{i-1}(M^0))$ (see [9], Sections 1 and 5, where the proofs are carried out for ideals). However, $Q'^0 = M^0(\mathfrak{p}) \cap L^0$ and $\mathrm{Cl}_{i-1}(M^0) = M'^0(\mathfrak{p}) \cap L^0$ and hence $Q'^0 : \mathrm{Cl}_{i-1}(M^0) = (M^0(\mathfrak{p}) \cap L^0) : (M'^0(\mathfrak{p}) \cap L^0) = (M^0(\mathfrak{p}) : (M'^0(\mathfrak{p}) \cap L^0)) \cap (L^0 : (M'^0(\mathfrak{p}) \cap L^0)) = M^0(\mathfrak{p}) : (M'^0(\mathfrak{p}) \cap L^0)$. From the fact that the associated primes of L are not contained in $\mathfrak{p}$, it follows that $M^0(\mathfrak{p}) : M'^0(\mathfrak{p}) \cap L^0 = M^0(\mathfrak{p}) : M'^0(\mathfrak{p}) = \mathfrak{e}^0(\mathfrak{p})$. Hence, $\pi_i^{\sigma_i}$ is the highest common factor of the polynomials contained in $P(u)[y_i, \cdots, y_n] \cap \mathfrak{e}^0(\mathfrak{p})$ which is the elementary divisor of $\mathfrak{e}^0(\mathfrak{p})$, i.e. of $\mathfrak{e}(\mathfrak{p})$ (see [9], Section 1). Since $\mathfrak{e}(\mathfrak{p})$ is $\mathfrak{p}$-primary and since $\rho(\mathfrak{p})$ is defined as the ordinary exponent of $\mathfrak{e}(\mathfrak{p})$, it follows from Theorem 7.1 of [3] that $\sigma_i = \rho(\mathfrak{p})$ in case P has characteristic zero.

The $\mathfrak{p}$-modal rank $\lambda(\mathfrak{p})$ of M and the $\mathfrak{p}$-modal exponent $\delta(\mathfrak{p})$ of M are essential for the theory of partial differential equations (see Part I).

LEMMA 2.62. *If the characteristic p of P is zero, the $\mathfrak{p}$-modal exponent $\delta(\mathfrak{p})$ is the smallest integer such that $M'(\mathfrak{p}) \cap \mathfrak{p}^{\delta(\mathfrak{p})} V \subseteq M(\mathfrak{p})$ and, if*

furthermore $\mathfrak{p}$ *is an isolated associated prime of* M, $\delta(\mathfrak{p}) = \rho(\mathfrak{p})$. *The* $\mathfrak{p}$*-modal rank* $\lambda(\mathfrak{p})$ *of* M *is, after a suitable linear transformation and for any characteristic* $p = 0$ *or* $\neq 0$, *equal to the* $\mathfrak{p}$*-degree* $a_0(\mathfrak{p})$ *of* M.

Proof. Using the notation of **1.2**, the ideals $\mathfrak{p}^{**}(\xi^{(1)}), \cdots, \mathfrak{p}^{**}(\xi^{(c)})$ are conjugated ideals with respect to the relative automorphisms of P^{**} with respect to P. Since the modules $M'^{**}(\mathfrak{p})$, $M^{**}(\mathfrak{p})$ and V^{**} are invariant under these relative automorphisms, $\delta(\mathfrak{p})$ could also have been defined in Definition 1.22 as the smallest integer such that

$$M'^{**}(\mathfrak{p}) \cap \left(\bigcap_{j=1}^{c} \mathfrak{p}^{**}(\xi^{(j)})\right)^{\delta(\mathfrak{p})} V^{**} \subseteq M^{**}(\mathfrak{p}).$$

Since the ideals $\mathfrak{p}^{**}(\xi^{(j)})$ are relatively prime,

$$\bigcap_{j=1}^{c} \mathfrak{p}^{**}(\xi^{(j)}) = \Pi_{j=1}^{c}\, \mathfrak{p}^{**}(\xi^{(j)});$$

and, since $p = 0$, $\Pi_{j=1}^{c}\mathfrak{p}^{**}(\xi^{(j)}) = \mathfrak{p}^{**}$ (see [7], Theorem 25). Hence, $\delta(\mathfrak{p})$ is the smallest integer such that $M'^{**}(\mathfrak{p}) \cap (\mathfrak{p}^{**})^{\delta(\mathfrak{p})} V^{**} \subseteq M^{**}(\mathfrak{p})$. Since the asterisks denote an algebraic extension of $\bar{P}$, it can easily be proved that they may be omitted for the definition of $\delta(\mathfrak{p})$. If $\mathfrak{p}$ is an isolated associated prime of M, $M'(\mathfrak{p}) = V$ and hence $\delta(\mathfrak{p})$ is the smallest integer such that $\mathfrak{p}^{\delta(\mathfrak{p})} M'(\mathfrak{p}) \subseteq M(\mathfrak{p})$, which means that $\delta(\mathfrak{p})$ is then the ordinary exponent of $\mathfrak{e}(\mathfrak{p})$, i. e. $\delta(\mathfrak{p}) = \rho(\mathfrak{p})$. The $\mathfrak{p}$-modal rank $\lambda(\mathfrak{p})$ is the P^{**}-rank of $M'^{**}(\mathfrak{p})/M^{**}(\mathfrak{p})$ and hence, according to Lemma 2.43, the $\bar{P}$-rank of $\overline{M'(\mathfrak{p})}/\overline{M(\mathfrak{p})}$. This rank is, according to Theorem 2.41, equal to la_0, where l is the length of $\overline{M(\mathfrak{p})}$ as a submodule of $\overline{M'(\mathfrak{p})}$ and a_0 is the $\bar{P}$-rank of $\bar{P}[x_1, \cdots, x_i]/\bar{\mathfrak{p}}$. Since $l = l(\mathfrak{p})$ and since, after a suitable linear transformation, a_0 is the degree of $\mathfrak{p}$, $\lambda(\mathfrak{p}) = a_0(\mathfrak{p})$ after a suitable linear transformation, which completes the proof.

2.7. General remarks. In the foregoing theory we have investigated those properties of a matrix $A = (\alpha_{ij})$, where $\alpha_{ij} \in P[x_1, \cdots, x_n]$, which are determined by the associated primes, the $M(\mathfrak{p})$, $M'(\mathfrak{p})$, $\mathfrak{e}(\mathfrak{p})$, $l(\mathfrak{p})$, $\rho(\mathfrak{p})$ and $a_0(\mathfrak{p})$, of the row space and column space of A. We shall refer to these ideals and integers as the invariants of A. If $n = 1$, these invariants can be computed from the sub-determinants of A (see [1]) and hence must be the same for the column space and the row space. The following example shows that the column space and row space of A may already have different associated primes when $n = 2$ and hence that the sub-determinants of A can not be used to compute the invariants of A.

Example 2.71. Consider the 2×2 matrix $A = (\alpha_{ij})$, where $\alpha_{ij} \in P[x, y]$ and where $\alpha_{11} = x$, $\alpha_{12} = 0$, $\alpha_{21} = y$, $\alpha_{22} = 0$. The column space is generated by the one column vector (x, y) and hence is a closed module which has only one associated prime, namely the zero ideal (ω) of $P[x, y]$. The row space M is generated by the row vectors $(x, 0)$ and $(y, 0)$. Hence $\mathrm{Cl}(M)$ is generated by the row vector $(1, 0)$ and it can be easily shown that M has two associated primes, namely $\mathfrak{p}_1 = (\omega)$ and $\mathfrak{p}_2 = (x, y)$.

It can be easily shown that, if $A = (\alpha_{ij})$ is an $m \times s$ matrix, $\alpha_{ij} \in P[x_1, \cdots, x_n]$, the invariants of A are invariant under similarity transformations PAQ where P and Q are square, invertible polynomial matrices. If $n = 1$, a Noether decomposition of the row space (or column space) of A gives rise to invariants, namely the classical elementary divisors of different rank, which determine A completely to within similarity transformation. Whether this is still the case for $n > 1$ is an unsolved problem. Equally unsolved, for $n > 1$, is the question whether invariants which arise from a Noether decomposition of a module $M \subseteq V$ determine the factor module V/M to within an isomorphism.

UNIVERSITY OF SOUTHERN CALIFORNIA,
LOS ANGELES, CALIFORNIA.

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